

# STRONG CONNECTIONS AND CHERN-CONNES PAIRING IN THE HOPF-GALOIS THEORY

LUDWIK DĄBROWSKI

Scuola Internazionale Superiore di Studi Avanzati

Via Beirut 2-4, 34014 Trieste, Italy

<http://www.sissa.it/fm/members/faculty/dabro.html>

HARALD GROSSE

Institute for Theoretical Physics, University of Vienna

Boltzmanngasse 5, 1090 Vienna, Austria

<http://www.thp.univie.ac.at/local/mathphys>

PIOTR M. HAJAC

Mathematical Institute, Polish Academy of Sciences

ul. Śniadeckich 8, Warsaw, 00-950 Poland

and

Department of Mathematical Methods in Physics

Warsaw University, ul. Hoża 74, Warsaw, 00-682 Poland

<http://www.fuw.edu.pl/~pmh>

## Abstract

We reformulate the concept of connection on a Hopf-Galois extension  $B \subseteq P$  in order to apply it in computing the Chern-Connes pairing between the cyclic cohomology  $HC^{2n}(B)$  and  $K_0(B)$ . This reformulation allows us to show that a Hopf-Galois extension admitting a strong connection is projective and left faithfully flat. It also enables us to conclude that a strong connection is a Cuntz-Quillen-type bimodule connection. To exemplify the theory, we construct a strong connection (super Dirac monopole) to find out the Chern-Connes pairing for the super line bundles associated to super Hopf fibration.

# Introduction

A noncommutative-geometric concept of principal bundles and characteristic classes is given by the Hopf-Galois theory of ring extensions and the pairing between cyclic cohomology and  $K$ -theory, respectively. In the spirit of the Serre-Swann theorem, the quantum vector bundles are given as finitely generated projective modules associated to an  $H$ -Galois extension via a corepresentation of Hopf algebra  $H$ . The  $K_0$ -class of such a module can be then paired with the cohomology class of a cyclic cocycle to produce an invariant playing the role of an integrated characteristic class of a vector bundle. To obtain these invariants, we provide a theory of connections on Hopf-Galois extensions which can be used in calculating projector matrices of associated quantum vector bundles. A main point of this paper is that strong connections on a Hopf-Galois extension  $B \subseteq P$  are equivalent to left  $B$ -linear right  $H$ -colinear unital splittings of the multiplication map  $B \otimes P \rightarrow P$ . Since connections can be considered as appropriate liftings of the translation map (restricted inverse of the canonical Galois map), knowing a connection yields automatically an *explicit* expression for the translation map. Vice-versa, an *explicit* formula for the translation map might immediately indicate a formula for connection. (This is important from the practical point of view.) If a connection is strong, then the simple machinery presented herein helps to extract the projective module data of an associated quantum vector bundle. One can then plug it in to the computation of the pairing. In the classical geometry, characteristic classes of associated vector bundles are computed from connections on *principal* bundles. Our approach parallels to some extent this idea in the quantum-geometric setting.

We work within the general framework of noncommutative geometry, quantum groups and Galois-type theories. For an introduction to Hopf-Galois extensions we refer to [M-S93, S-HJ94] and for a comprehensive description of the Chern-Connes pairing to [C-A94, L-JL97]. The point of view advocated in here was already employed to compute projector matrices [HM99] and the Chern numbers [H-PM] of the quantum Hopf line bundles from the Dirac  $q$ -monopole connection [BM93]. Thus, although this work is antedated by [HM99] and [H-PM], it conceptually precedes these papers, and can be viewed as a follow up of the theory of connections, strong connections and associated quantum vector bundles developed in [BM93], [H-PM96] and [D-M96], respectively. (See [D-M97a, Section 5] and [D-M97c, D-Ma] for an alternative theory of characteristic classes on quantum principal bundles.)

We begin in Section 1 by recalling basic facts and definitions. In Section 2 we first reformulate the concept of general connections so as to make transparent the characterisation of a strong connection as an appropriate splitting of the multiplication map  $B \otimes P \rightarrow P$ , where  $P$  is an  $H$ -Galois extension of  $B$ . Then we prove the equivalence of four different definitions of a strong connection, which is the main claim of this paper, and study its consequences. As a quick illustration of the theory, we apply it to a strong and non-strong connection on quantum projective space  $\mathbb{R}P_q^2$ . We obtain, as a by-product, the definition of the “tangent bundle” of the Podleś equator quantum sphere. We also show that there are infinitely many canonical strong connections on the quantum Hopf fibration, and prove that they all coincide with the Dirac monopole in the classical limit. A super Dirac monopole is presented in Section 3. We adapt to the Hopf-Galois setting the construction of a super Hopf fibration. Then, employing the super monopole, we compute projector matrices of the super Hopf line bundles. Taking advantage of the functoriality of the Chern-Connes pairing, we conclude that the value of the

pairing for the super and classical Hopf line bundles coincides. Hence we infer the non-cleanness of the super Hopf fibration. We end Section 3 by proving that, in analogy with the classical situation, the direct sum of spin-bundle modules (Dirac spinors) is free of rank two for both the super and the quantum Hopf fibration. In Appendix, we complement the four descriptions of a strong connection by providing (appropriately adapted) four equivalent actions of gauge transformations on connections.

## 1 Preliminaries

Throughout the paper algebras are assumed to be unital and over a field  $k$ . The unadorned tensor product stands for the tensor product over  $k$ . Our approach is algebraic, so that we use finite sums. We use the Sweedler notation  $\Delta h = h_{(1)} \otimes h_{(2)}$  (summation understood) and its derivatives. The letter  $S$  and  $\varepsilon$  signify the antipode and counit, respectively. The convolution product of two linear maps from a coalgebra to an algebra is denoted in the following way:  $(f * g)(c) := f(c_{(1)})g(c_{(2)})$ . We use the word “colinear” with respect to linear maps that preserve the comodule structure. (Such maps are also called “covariant.”) We work with right Hopf-Galois extensions and skip writing “right” for brevity. For an  $H$ -Galois extension  $B \subseteq P$  we write the canonical Galois isomorphism as

$$\chi := (m \otimes \text{id}) \circ (\text{id} \otimes_B \Delta_R) : P \otimes_B P \longrightarrow P \otimes H , \quad (1.1)$$

where  $\Delta_R : P \rightarrow P \otimes H$  stands for the comodule algebra coaction ( $\Delta_R p =: p_{(0)} \otimes p_{(1)}$ ; again, summation understood), and  $m$  for the multiplication map  $P \otimes P \rightarrow P$ . We say that a Hopf-Galois extension is cleft iff there exists a (unital) convolution-invertible colinear map  $\Phi : H \rightarrow P$ , and call  $\Phi$  a cleaving map. The concept of cleanness is close but, as explained in the last paragraph of [DHS99, Section 4], not tantamount to the idea of triviality of a principal bundle. (Trivial is cleft but not vice-versa.) A cleaving map is usually assumed to be unital, but since any non-unital  $\Phi$  can be unitalised (e.g., see [DT86, p.813] or [HM99, Section 1]), this assumption, though technically useful, is conceptually redundant. It also follows from the defining properties of  $\Phi$  that it is injective (e.g., see [HM99, Section 1]).

Next, note that the canonical map  $\chi$ , although cannot be an algebra homomorphism in general, is always determined by its values on generators. The same is true for  $\chi^{-1}$ . The left  $P$ -linearity of  $\chi^{-1}$  makes it practical to restrict it from  $P \otimes H$  to  $H$ , and define the translation map

$$\tau : H \rightarrow P \otimes_B P , \quad \tau(h) := \chi^{-1}(1 \otimes h) =: h^{[1]} \otimes_B h^{[2]} \quad (\text{summation understood}). \quad (1.2)$$

The following are properties of  $\tau$  compiled from [S-HJ90b, B-T96]:

$$(\text{id} \otimes_B \Delta_R) \circ \tau = (\tau \otimes \text{id}) \circ \Delta , \quad (1.3)$$

$$((\text{flip} \circ \Delta_R) \otimes_B \text{id}) \circ \tau = (S \otimes \tau) \circ \Delta , \quad (1.4)$$

$$\Delta_{P \otimes_B P} \circ \tau = (\tau \otimes \text{id}) \circ \text{Ad}_R , \quad (1.5)$$

$$m \circ \tau = \varepsilon , \quad (1.6)$$

$$\tau(h\tilde{h}) = \tilde{h}^{[1]}h^{[1]} \otimes_B h^{[2]}\tilde{h}^{[2]} . \quad (1.7)$$

Here  $\Delta_{P \otimes_B P}$  is the coaction on  $P \otimes_B P$  obtained via the canonical surjection  $\pi_B : P \otimes P \rightarrow P \otimes_B P$  from the diagonal coaction

$$\Delta_{P \otimes P} : p \otimes p' \mapsto p_{(0)} \otimes p'_{(0)} \otimes p_{(1)} p'_{(1)} , \quad (1.8)$$

and  $\text{Ad}_R(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$  is the right adjoint coaction.

To fix convention and clarify some basic issues, let us recall that the universal differential calculus  $\Omega^1 A$  (grade one of the universal differential algebra) can be defined by the exact sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0 , \quad (1.9)$$

i.e., as the kernel of the multiplication map. The differential is given by  $da := 1 \otimes a - a \otimes 1$ . We can identify  $\Omega^1 A$  with  $A \otimes A/k$  as left  $A$ -modules via the maps

$$\Omega^1 A \ni \sum_i a_i \otimes a'_i \mapsto \sum_i a_i \otimes \pi_A(a'_i) \in A \otimes A/k \ni x \otimes \pi_A(y) \mapsto xdy \in \Omega^1 A , \quad (1.10)$$

where  $\pi_A : A \rightarrow A/k$  is the canonical surjection. Similarly, one can identify  $\Omega^1 A$  with  $A/k \otimes A$  as right  $A$ -modules ( $\sum_i a_i \otimes a'_i \mapsto \sum_i \pi_A(a_i) \otimes a'_i$ ). Consequently, for any left  $A$ -module  $N$ , we have  $\Omega^1 A \otimes_A N \cong A/k \otimes N$ . For any splitting  $\iota : A/k \rightarrow A$  of the canonical surjection ( $\pi_A \circ \iota = \text{id}$ ), we have an injection  $\iota \otimes \text{id} : A/k \otimes N \rightarrow A \otimes N$ . Thus there is an injection

$$f_\iota : \Omega^1 A \otimes_A N \longrightarrow A \otimes N , \quad f_\iota \left( \sum_{i,j} a_{ij} \otimes a'_{ij} \otimes_A n_j \right) := \sum_{i,j} (\iota \circ \pi_A)(a_{ij}) \otimes a'_{ij} n_j . \quad (1.11)$$

On the other hand, we have a natural map coming from tensoring (1.9) on the right with  $N$ :

$$f_N : \Omega^1 A \otimes_A N \longrightarrow A \otimes N , \quad f_N \left( \sum_{i,j} a_{ij} \otimes a'_{ij} \otimes_A n_j \right) := \sum_{i,j} a_{ij} \otimes a'_{ij} n_j . \quad (1.12)$$

Since  $\pi_A \circ \iota = \text{id}$ , we have  $(\pi_A \otimes \text{id}) \circ f_N = (\pi_A \otimes \text{id}) \circ f_\iota$ , whence

$$((\iota \circ \pi_A) \otimes \text{id}) \circ f_N = ((\iota \circ \pi_A) \otimes \text{id}) \circ f_\iota = f_\iota . \quad (1.13)$$

It follows now from the injectivity of  $f_\iota$  that  $f_N$  is injective. Thus we have shown that (1.9) yields the exact sequence:

$$0 \longrightarrow \Omega^1 A \otimes_A N \longrightarrow A \otimes N \longrightarrow N \longrightarrow 0 . \quad (1.14)$$

If  $B$  is a subalgebra of  $P$ , then we can also write  $(\Omega^1 B)P$  for the kernel of the multiplication map  $B \otimes P \rightarrow P$ . Indeed,  $m((\Omega^1 B)P) = 0$ , and if  $\sum_i b_i \otimes p_i \in \text{Ker}(B \otimes P \xrightarrow{m} P)$ , then

$$\sum_i b_i \otimes p_i = \sum_i (b_i \otimes p_i - 1 \otimes b_i p_i) = - \sum_i (db_i) p_i \in (\Omega^1 B)P . \quad (1.15)$$

To sum up, we have (cf. [HM99, p.251])

$$\Omega^1 B \otimes_B P \cong \text{Ker}(B \otimes P \xrightarrow{m} P) = (\Omega^1 B)P . \quad (1.16)$$

The following are the universal-differential-calculus versions of general-calculus definitions in [BM93, H-PM96]:

**Definition 1.1** ([BM93]) *Let  $B \subseteq P$  be an  $H$ -Galois extension. Denote by  $\Omega^1 P$  the universal differential calculus on  $P$  and by  $\Delta_{\Omega^1 P}$  the restriction of  $\Delta_{P \otimes P}$  to  $\Omega^1 P$ . A left  $P$ -module projection  $\Pi$  on  $\Omega^1 P$  is called a connection iff*

$$\text{Ker } \Pi = P(\Omega^1 B)P \quad (\text{horizontal forms}), \quad (1.17)$$

$$\Delta_{\Omega^1 P} \circ \Pi = (\Pi \otimes \text{id}) \circ \Delta_{\Omega^1 P} \quad (\text{right colinearity}). \quad (1.18)$$

**Definition 1.2** ([BM93]) *Let  $P$ ,  $H$ ,  $B$  and  $\Omega^1 P$  be as above. A  $k$ -homomorphism  $\omega : H \rightarrow \Omega^1 P$  such that  $\omega(1) = 0$  is called a connection form iff it satisfies the following properties:*

1.  $(m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) \circ \omega = 1 \otimes (\text{id} - \varepsilon)$  (fundamental vector field condition),
2.  $\Delta_{\Omega^1 P} \circ \omega = (\omega \otimes \text{id}) \circ \text{Ad}_R$  (right adjoint colinearity).

For every Hopf-Galois extension there is a one-to-one correspondence between connections and connection forms (see [BM93, p.606] or [M-S97, Proposition 2.1]). In particular, the connection  $\Pi^\omega$  associated to a connection form  $\omega$  is given by the formula:

$$\Pi^\omega(dp) = p_{(0)}\omega(p_{(1)}). \quad (1.19)$$

(Since  $\Pi^\omega$  is a left  $P$ -module homomorphism, it suffices to know its values on exact forms.)

**Definition 1.3** ([H-PM96]) *Let  $\Pi$  be a connection in the sense of Definition 1.1. It is called strong iff  $(\text{id} - \Pi)(dP) \subseteq (\Omega^1 B)P$ . We say that a connection form is strong iff its associated connection is strong.*

Let us now have a closer look at the concept of connection. For the sake of brevity we put

$$\bar{\chi} = (m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) : P \otimes P \rightarrow P \otimes H, \quad (1.20)$$

denote by  $\hat{\chi}$  its restriction to  $\Omega^1 P$ , and by  $H^+$  the kernel of the counit map (augmentation ideal). Since  $((\text{id} \otimes \varepsilon) \circ \bar{\chi})(\Omega^1 P) = 0$ , we have  $\hat{\chi}(\Omega^1 P) = P \otimes H^+$ . Consider  $P \otimes H$ , and similarly  $P \otimes H^+$ , as a right comodule via the map

$$\Delta_{P \otimes H} : p \otimes h \longmapsto p_{(0)} \otimes h_{(2)} \otimes p_{(1)} S(h_{(1)}) h_{(3)}. \quad (1.21)$$

Then there is a one-to-one correspondence between connections and left  $P$ -linear right  $H$ -colinear splittings of  $\hat{\chi}$  [BM93, p.606]. Since  $H = H^+ \oplus k$ , we can define

$$\bar{\sigma}(p \otimes h) = \begin{cases} \sigma(p \otimes h) & \text{for } h \in H^+ \\ p \otimes h 1_P & \text{for } h \in k, \end{cases} \quad (1.22)$$

where  $\sigma$  is a splitting of  $\hat{\chi}$ . On the other hand, we can consider unital left  $P$ -linear right  $H$ -colinear splittings  $r$  of the canonical surjection  $\pi_B : P \otimes P \rightarrow P \otimes_B P$ . This leads to the following commutative diagram of exact rows of left  $P$ -modules right  $H$ -comodules (see above for the comodule structures):

$$\begin{array}{ccccccc}
0 & \longrightarrow & P(\Omega^1 B)P & \longrightarrow & \Omega^1 P & \begin{array}{c} \xrightarrow{\hat{\chi}} \\ \xleftarrow{\sigma} \end{array} & P \otimes H^+ \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & P(\Omega^1 B)P & \longrightarrow & P \otimes P & \begin{array}{c} \xrightarrow{\bar{\chi}} \\ \xleftarrow{\bar{\sigma}} \end{array} & P \otimes H \longrightarrow 0 \\
& & \parallel & & \parallel & & \downarrow \chi^{-1} \\
0 & \longrightarrow & P(\Omega^1 B)P & \longrightarrow & P \otimes P & \begin{array}{c} \xrightarrow{\pi_B} \\ \xleftarrow{\tau} \end{array} & P \otimes_B P \longrightarrow 0.
\end{array} \tag{1.23}$$

One can check that  $\bar{\chi}$  intertwines the relevant comodule structures

$$\begin{aligned}
(\Delta_{P \otimes H} \circ \bar{\chi})(p \otimes p') &= \Delta_{P \otimes H}(pp'_{(0)} \otimes p'_{(1)}) \\
&= p_{(0)}p'_{(0)} \otimes p'_{(3)} \otimes p_{(1)}p'_{(1)}S(p'_{(2)})p'_{(4)} \\
&= p_{(0)}p'_{(0)} \otimes p'_{(1)} \otimes p_{(1)}p'_{(2)} \\
&= (\bar{\chi} \otimes \text{id})(p_{(0)} \otimes p'_{(0)} \otimes p_{(1)}p'_{(1)}) \\
&= ((\bar{\chi} \otimes \text{id}) \circ \Delta_{P \otimes P})(p \otimes p').
\end{aligned} \tag{1.24}$$

Other calculations to verify that this diagram is a commutative diagram of right  $H$ -comodules are of the same kind. To see that  $\text{Ker } \pi_B = P(\Omega^1 B)P$  one can argue as above (1.16).

Yet another description of a connection as a splitting is as follows. Denote  $\pi_B(\Omega^1 P)$  by  $\Omega_B^1 P$  (relative differential forms as in [CQ95, Section 2]). The commutativity of the diagram (1.23) implies that the restriction of the canonical map  $\tilde{\chi} : \Omega_B^1 P \rightarrow P \otimes H^+$  is an isomorphism. Let  $\omega$  be a connection form and  $\tilde{\omega}$  its restriction to  $H^+$ . Similarly, let  $\tilde{\tau}$  be the restriction to  $H^+$  of the translation map. Recall that  $\sigma$  is the left  $P$ -module extension of  $\tilde{\omega}$  [BM93, p.606]. Hence the commutativity of (1.23) implies also (for any  $\tilde{\omega}$ )  $\pi_B \circ \tilde{\omega} = \tilde{\tau}$ . Since the translation map  $\tau$  is unital, knowing  $\tilde{\tau}$  is tantamount to knowing  $\tau$ . Thus a connection form yields an *explicit* expression for the translation map. On the other hand, viewing  $H^+$  as a right comodule under the right adjoint coaction  $\text{Ad}_R$  allows one to define equivalently a connection as a *colinear* lifting of the restricted translation map  $\tilde{\tau}$ . Indeed, we can complete the equality  $\pi_B \circ \tilde{\omega} = \tilde{\tau}$  to the commutative diagram

$$\begin{array}{ccc}
\Omega^1 P & \xrightarrow{\hat{\chi}} & P \otimes H^+ \\
\uparrow \tilde{\omega} & \searrow \pi_B & \uparrow \tilde{\chi} \\
H^+ & \xrightarrow{\tilde{\tau}} & \Omega_B^1 P
\end{array} \tag{1.25}$$

and directly verify this assertion. This explains the close resemblance between the formulas for the translation maps and connection forms. For example, compare (3.5) with (3.10-3.11) and the proof of [H-PM96, Proposition 2.10] with [H-PM96, 2.14]. Compare also [DHS99, Corollary 2.3] and [BM93, Proposition 5.3].

A natural next step is to consider associated quantum vector bundles. More precisely, what we need here is a replacement of the module of sections of an associated vector bundle. In the classical case such sections can be equivalently described as “functions of type  $\varrho$ ” from the total space of a principal bundle to a vector space. We follow this construction in the

quantum case by considering  $B$ -bimodules of colinear maps  $\text{Hom}_\rho(V, P)$  associated with an  $H$ -Galois extension  $B \subseteq P$  via a corepresentation  $\rho : V \rightarrow V \otimes H$  (see [D-M97a, Appendix B] or [D-M96]).

Proposition 2.5 gives a formula for a splitting of the multiplication map  $B \otimes \text{Hom}_\rho(V, P) \rightarrow \text{Hom}_\rho(V, P)$ , and a splitting of the multiplication map is almost the same as a projector matrix, for it is an embedding of  $\text{Hom}_\rho(V, P)$  in the free  $B$ -module  $B \otimes \text{Hom}_\rho(V, P)$ . However, to turn a splitting into a concrete recipe for producing finite size projector matrices of finitely generated projective modules, we need the following general lemma:

**Lemma 1.4 ([HM99])** *Let  $A$  be an algebra and  $M$  a projective left  $A$ -module generated by linearly independent generators  $g_1, \dots, g_n$ . Also, let  $\{\tilde{g}_\mu\}_{\mu \in I}$  be a completion of  $\{g_1, \dots, g_n\}$  to a linear basis of  $M$ ,  $f_2$  be a left  $A$ -linear splitting of the multiplication map  $A \otimes M \rightarrow M$  given by the formula  $f_2(g_k) = \sum_{l=1}^n a_{kl} \otimes g_l + \sum_{\mu \in I} a_{k\mu} \otimes \tilde{g}_\mu$ , and  $c_{\mu l} \in A$  a choice of coefficients such that  $\tilde{g}_\mu = \sum_{l=1}^n c_{\mu l} g_l$ . Then  $E_{kl} = a_{kl} + \sum_{\mu \in I} a_{k\mu} c_{\mu l}$  defines a projector matrix of  $M$ , i.e.,  $E \in M_n(A)$ ,  $E^2 = E$  and  $A^n E$  (row times matrix) and  $M$  are isomorphic as left  $A$ -modules.*

For our later purpose, we also need the following general digression. Let  $A$  be an algebra, and let  $E, F$  be idempotents in  $M_m(A)$ ,  $M_n(A)$ , respectively. It can be verified that the projective modules  $A^m E$  and  $A^n F$  are isomorphic if there exist maps  $L$  and  $\tilde{L}$

$$\begin{array}{ccc} A^m & \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{\tilde{L}} \end{array} & A^n \\ \downarrow \uparrow & & \downarrow \uparrow \\ A^m E & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & A^n F \end{array} \quad (1.26)$$

such that

$$ELF = LF, \quad F\tilde{L}E = \tilde{L}E, \quad EL\tilde{L} = E, \quad F\tilde{L}L = F. \quad (1.27)$$

## 2 Connections

We begin this section by considering general connections on Hopf-Galois extensions as appropriate splittings. It is known that, under the assumption of faithful flatness, there always exist a connection on a Hopf-Galois extension [S-P93, Satz 6.3.5] (cf. [D-M97a, Theorem 4.1]). (For a comprehensive review of faithful flatness see [B-N72].) Chasing diagram (1.23) and playing around with appropriate modifications of its rows we obtain:

**Proposition 2.1** *Let  $B \subseteq P$  be an  $H$ -Galois extension. Denote by  $\mathcal{C}(P)$  the space of connection forms on  $P$ , by  $\mathcal{R}(P)$  the space of unital left  $P$ -linear right  $H$ -colinear splittings  $r$  of the canonical surjection  $\pi_B : P \otimes P \rightarrow P \otimes_B P$ , and by  $\mathcal{S}(P)$  the space of unital left  $B$ -linear right  $H$ -colinear maps  $s : P \rightarrow P \otimes P$  satisfying  $(\pi_B \circ s)(p) = 1 \otimes_B p$ . Then the formulas*

$$\Psi(\omega)(p \otimes_B p') = pp' \otimes 1 + pp'_{(0)} \omega(p'_{(1)}), \quad \tilde{\Psi}(r)(h) = (r \circ \tau)(h - \varepsilon(h)), \quad (2.1)$$

*define mutually inverse bijections  $\mathcal{C}(P) \xrightarrow{\Psi} \mathcal{R}(P) \xrightarrow{\tilde{\Psi}} \mathcal{C}(P)$  and, similarly, the formulas*

$$\Xi(r)(p) = r(1 \otimes_B p), \quad \tilde{\Xi}(s)(p \otimes_B p') = ps(p'). \quad (2.2)$$

determine mutually inverse bijections  $\mathcal{R}(P) \xrightarrow{\Xi} \mathcal{S}(P) \xrightarrow{\tilde{\Xi}} \mathcal{R}(P)$ .

*Proof.* Let us first check that  $\Psi(\mathcal{C}(P)) \subseteq \mathcal{R}(P)$ . It is clear that, for any  $\omega \in \mathcal{C}(P)$ ,  $\Psi(\omega)$  is unital and left  $P$ -linear. To see that  $\Psi(\omega)$  is  $H$ -colinear, we use (1.19) and (1.18):

$$\begin{aligned} (\Delta_{P \otimes P} \circ \Psi(\omega))(p \otimes_B p') &= \Delta_{P \otimes P}(pp' \otimes 1 + \Pi^\omega(pd p')) \\ &= (p_{(0)}p'_{(0)} \otimes 1 + \Pi^\omega(p_{(0)}dp'_{(0)})) \otimes p_{(1)}p'_{(1)} \\ &= \Psi(\omega)(p_{(0)} \otimes_B p'_{(0)}) \otimes p_{(1)}p'_{(1)} \\ &= ((\Psi(\omega) \otimes \text{id}) \circ \Delta_{P \otimes_B P})(p \otimes_B p'). \end{aligned} \quad (2.3)$$

To verify that  $\Psi(\omega)$  is a splitting of the canonical surjection  $\pi_B$ , recall that  $\text{Ker } \pi_B = P(\Omega^1 B)P$  and note that  $(\Pi^\omega)^2 = \Pi^\omega$  entails  $\text{Ker } \Pi^\omega = (\text{id} - \Pi^\omega)(\Omega^1 P)$ . Thus, by (1.17), we have  $\text{Ker } \pi_B = (\text{id} - \Pi^\omega)(\Omega^1 P)$ . Combining this with (1.19) we obtain

$$\begin{aligned} (\text{id} - \pi_B \circ \Psi(\omega))(p \otimes_B p') &= \pi_B(p \otimes p' - pp' \otimes 1 - \Pi^\omega(pd p')) \\ &= (\pi_B \circ (\text{id} - \Pi^\omega))(pd p') = 0. \end{aligned} \quad (2.4)$$

The next step is to check that  $\tilde{\Psi}(\mathcal{R}(P)) \subseteq \mathcal{C}(P)$ . To see that  $\tilde{\Psi}(r)(H) \subseteq \Omega^1 P$  for any  $r \in \mathcal{R}(P)$ , we take advantage of property (1.6) of the translation map  $\tau$ , and compute:

$$(m \circ \tilde{\Psi}(r))h = (m \circ \pi_B \circ r \circ \tau)(h - \varepsilon(h)) = (m \circ \tau)(h - \varepsilon(h)) = \varepsilon(h - \varepsilon(h)) = 0. \quad (2.5)$$

(Here we abuse the notation and denote also by  $m$  the multiplication map on  $P \otimes_B P$ .) It is immediate that  $\tilde{\Psi}(r)(1) = 0$ . Furthermore, using the colinearity of  $r$  and (1.6), we verify the colinearity of  $\tilde{\Psi}(r)$ :  $\Delta_{\Omega^1 P} \circ \tilde{\Psi}(r)(h) = (\omega \otimes \text{id}) \circ \text{Ad}_R$ . To check the fundamental-vector-field condition we note

$$(\bar{\chi} \circ r \circ \tau)(h - \varepsilon(h)) = 1 \otimes (h - \varepsilon(h)), \quad (2.6)$$

which is equivalent to  $(\bar{\chi} \circ r \circ \chi^{-1}) = \text{id}$ . The latter equality, however, follows from the commutativity of (1.23), as needed.

It remains to show that  $\tilde{\Psi} \circ \Psi = \text{id}$  and  $\Psi \circ \tilde{\Psi} = \text{id}$ . To this end, taking advantage of the unitality of  $\Psi(\omega)$ , (1.3) and (1.6), we compute:

$$\begin{aligned} ((\tilde{\Psi} \circ \Psi)(\omega))(h) &= (\Psi(\omega) \circ \tau)(h - \varepsilon(h)) \\ &= \Psi(\omega)(h^{[1]} \otimes_B h^{[2]}) - \varepsilon(h) \otimes 1 \\ &= \varepsilon(h) \otimes 1 + h^{[1]}h^{[2]}_{(0)}\omega(h^{[2]}_{(1)}) - \varepsilon(h) \otimes 1 \\ &= h_{(1)}^{[1]}h_{(1)}^{[2]}\omega(h_{(2)}) \\ &= \varepsilon(h_{(1)})\omega(h_{(2)}) = \omega(h). \end{aligned} \quad (2.7)$$

Similarly, taking advantage of the unitality and left  $P$ -linearity of  $r$ , we compute

$$\begin{aligned} ((\Psi \circ \tilde{\Psi})(r))(p \otimes_B p') &= pp' \otimes 1 + pp'_{(0)}(\tilde{\Psi}(r))(p'_{(1)}) \\ &= pp' \otimes 1 + pp'_{(0)}(r \circ \tau)(p'_{(1)} - \varepsilon(p'_{(1)})) \\ &= pp' \otimes 1 + r(pp'_{(0)}\tau(p'_{(1)})) - pp'_{(0)}\varepsilon(p'_{(1)}) \otimes 1 \\ &= r(\chi^{-1}(\chi(p \otimes_B p'))) = r(p \otimes_B p'). \end{aligned} \quad (2.8)$$

Finally, the proof concerning  $\Xi$  and  $\tilde{\Xi}$  is straightforward.  $\square$



**Corollary 2.2** *An  $H$ -Galois extension  $B \subseteq P$  admits a connection, if there exists a (not necessarily unital) left  $B$ -linear right  $H$ -colinear map  $s : P \rightarrow P \otimes P$  satisfying  $(\pi_B \circ s)(p) = 1 \otimes_B p$ .*

*Proof.* Denote by  $\overline{\mathcal{S}(P)}$  the space of all maps  $s$  defined in the corollary. To construct a “unitalising” map  $\mathcal{T} : \overline{\mathcal{S}(P)} \rightarrow \mathcal{S}(P)$ , note that, for  $\bar{s} \in \overline{\mathcal{S}(P)}$  and  $s \in \mathcal{S}(P)$ , we must have  $s(1) - \bar{s}(1) = 1 \otimes 1 - \bar{s}(1)$ . Extending this equality by the left  $P$ -linearity, we can define

$$\mathcal{T}(\bar{s})(p) = \bar{s}(p) + p(1 \otimes 1 - \bar{s}(1)). \quad (2.9)$$

It is straightforward to check that  $\mathcal{T}(\overline{\mathcal{S}(P)}) \subseteq \mathcal{S}(P)$ , as needed.  $\square$

When we think of a connection as an element  $s \in \mathcal{S}(P)$ , then the strongness condition (see Definition 1.3) can be put as  $s(P) \subseteq B \otimes P$ . (Shift the second term on the right hand side to the left hand side in [M-S97, (11)].) Describing strong connections as strong elements in  $\mathcal{S}(P)$  is the main point of the below theorem. The second description is in terms of a covariant differential, and was hinted at in [H-PM96, Remark 4.3]. The third one coincides with the definition of a strong connection except that we change the inclusion condition  $(\text{id} - \Pi)(dP) \subseteq (\Omega^1 B)P$  to the equivalent equality condition  $(\text{id} - \Pi)(BdP) = (\Omega^1 B)P$ . The last description is precisely the definition of a strong connection form. Thus there is no essentially new approach to connections in the following theorem. However, proving the equivalence of a strong connection to an appropriate splitting of the multiplication map  $B \otimes P \rightarrow P$  enables us to derive several desirable consequences. We write everything explicitly so as to provide a self-contained and coherent treatment of the strong connection.

**Theorem 2.3** *Let  $B \subseteq P$  be an  $H$ -Galois extension. The following are equivalent descriptions of a strong connection:*

- 1) *A unital left  $B$ -linear right  $H$ -colinear splitting  $s$  of the multiplication map  $B \otimes P \xrightleftharpoons[s]{m} P$ .*
- 2) *A right  $H$ -colinear homomorphism  $D : P \rightarrow (\Omega^1 B)P$  annihilating 1 and satisfying the Leibniz rule:  $D(bp) = bDp + db.p$ ,  $\forall b \in B, p \in P$ .*
- 3) *A left  $P$ -linear right  $H$ -colinear projection  $\Pi : \Omega^1 P \rightarrow \Omega^1 P$  ( $\Pi^2 = \Pi$ ) such that  $(\text{id} - \Pi)(BdP) = (\Omega^1 B)P$ .*
- 4) *A homomorphism  $\omega : H \rightarrow \Omega^1 P$  vanishing on 1 and satisfying:*
  - a)  $\Delta_{\Omega^1 P} \circ \omega = (\omega \otimes \text{id}) \circ \text{Ad}_R$
  - b)  $(m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) \circ \omega = 1 \otimes (\text{id} - \varepsilon)$
  - c)  $dp - p_{(0)}\omega(p_{(1)}) \in (\Omega^1 B)P$ ,  $\forall p \in P$ .

*Proof.* Let  $V_i, i \in \{1, 2, 3, 4\}$ , denote the corresponding spaces of homomorphisms defined in points 1)–4). We need to construct 4 mappings

$$J_1 : V_1 \rightarrow V_2, \quad J_2 : V_2 \rightarrow V_3, \quad J_3 : V_3 \rightarrow V_4, \quad J_4 : V_4 \rightarrow V_1, \quad (2.10)$$

satisfying 4 identities:

$$J_4 \circ J_3 \circ J_2 \circ J_1 = \text{id} \quad \text{and cyclicly permuted versions.} \quad (2.11)$$

Put  $J_1(s)(p) = 1 \otimes p - s(p)$ . (Compare with the right-handed version [CQ95, (55)].) Evidently,  $J_1(s)$  is a right  $H$ -colinear homomorphism from  $P$  to  $(\Omega^1 B)P = \text{Ker}(m : B \otimes P \rightarrow P)$  (see (1.16)) annihilating 1. As for the Leibniz rule, we have

$$J_1(s)(bp) = 1 \otimes bp - s(bp) = db.p + b \otimes p - bs(p) = db.p + bJ_1(s)(p). \quad (2.12)$$

This establishes  $J_1$  as a map from  $V_1$  to  $V_2$ .

Next, put  $J_2(D)(p'dp) = p'(d - D)(p)$ . Observe first that  $J_2(D)$  is a well-defined endomorphism of  $\Omega^1 P$  because  $D1 = 0$  (see (1.10)). Choose  $b_i \in B$ ,  $p_i \in P$ , such that  $Dp = \sum_i (db_i)p_i = \sum_i (d(b_i p_i) - b_i dp_i)$ . It follows from the Leibniz rule that  $J_2(D) \circ d$  is a left  $B$ -module map. Thus we have:

$$\begin{aligned} J_2(D)^2(p'dp) &= J_2(D)(p'dp) - J_2(D)(p'Dp) \\ &= J_2(D)(p'dp) - \sum_i p' J_2(D)(db_i.p_i) \\ &= J_2(D)(p'dp) - \sum_i p'(J_2(D) \circ d)(b_i p_i) + \sum_i p' b_i (J_2(D) \circ d)(p_i) \\ &= J_2(D)(p'dp), \end{aligned} \quad (2.13)$$

Hence  $J_2(D)$  is a projection. Furthermore, note that for any  $b_i \in B$ ,  $p_i \in P$ , we have

$$(\text{id} - J_2(D))(\sum_i b_i . dp_i) = \sum_i b_i . Dp_i \in (\Omega^1 B)P, \quad (2.14)$$

i.e.,  $(\text{id} - J_2(D))(\sum_i B dP) \subseteq (\Omega^1 B)P$ . To see the reverse inclusion, take any  $\sum_i b_i \otimes p_i \in \text{Ker}(B \otimes P \xrightarrow{m} P) = (\Omega^1 B)P$ . Then, using the above calculation and the Leibniz rule, we obtain

$$0 = \sum_i D(b_i p_i) = \sum_i b_i Dp_i + \sum_i db_i . p_i = (\text{id} - J_2(D))(\sum_i b_i . dp_i) - \sum_i b_i \otimes p_i, \quad (2.15)$$

i.e.,  $\sum_i b_i \otimes p_i \in \text{Im}(\text{id} - J_2(D))$ , as needed.

To construct  $J_3$ , note first that  $(\Pi \circ d) : P \rightarrow \Omega^1 P$  is left  $B$ -linear. Indeed, since  $\Pi^2 = \Pi$ , the condition  $(\text{id} - \Pi)(BdP) = (\Omega^1 B)P$  entails  $\Pi((\Omega^1 B)P) = 0$ . Consequently

$$\Pi d(bp) = \Pi(db.p) + \Pi(bdp) = b(\Pi \circ d)(p), \quad (2.16)$$

as claimed. Therefore it makes sense to put  $J_3(\Pi)(h) = h^{[1]}\Pi(dh^{[2]})$  (see (1.2)). This formula defines a homomorphism from  $H$  to  $\Omega^1 P$  vanishing on 1. Furthermore, by the right  $H$ -colinearity of  $\Pi$  and property (1.5) of the translation map, we have

$$\begin{aligned} (\Delta_{\Omega^1 P} \circ J_3(\Pi))(h) &= h^{[1]}_{(0)} \Pi(dh^{[2]}_{(0)}) \otimes h^{[1]}_{(1)} h^{[2]}_{(1)} \\ &= h^{[1]}_{(2)} \Pi(dh^{[2]}_{(2)}) \otimes S(h_{(1)}) h_{(3)} \\ &= ((J_3(\Pi) \otimes \text{id}) \circ \text{Ad}_R)(h). \end{aligned} \quad (2.17)$$

As for the property b), note first that  $(\text{id} - \Pi)(BdP) = (\Omega^1 B)P$  implies, by the left  $P$ -linearity of  $\Pi$ , that  $(\text{id} - \Pi)(\Omega^1 P) = P(\Omega^1 B)P$ . Secondly, recall that  $\overline{\chi}(P(\Omega^1 B)P) = 0$  (see (1.23)). Hence

$$\begin{aligned} (\overline{\chi} \circ J_3(\Pi))(h) &= h^{[1]}((m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R))(\Pi d h^{[2]} + (\text{id} - \Pi) d h^{[2]}) \\ &= h^{[1]}((m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R))(1 \otimes h^{[2]} - h^{[2]} \otimes 1) \\ &= h^{[1]} h^{[2]}_{(0)} \otimes h^{[2]}_{(1)} - h^{[1]} h^{[2]} \otimes 1 \\ &= 1 \otimes (h - \varepsilon(h)). \end{aligned} \quad (2.18)$$

To verify c), we compute:

$$dp - p_{(0)}p_{(1)}^{[1]}\Pi dp_{(1)}^{[2]} = dp - \Pi dp = (\text{id} - \Pi)dp \in (\Omega^1 B)P. \quad (2.19)$$

Consequently,  $J_3$  is a mapping from  $V_3$  to  $V_4$ .

Finally, put

$$J_4(\omega)(p) = p \otimes 1 + p_{(0)}\omega(p_{(1)}) . \quad (2.20)$$

To see that  $J_4(\omega)$  takes values in  $B \otimes P$ , note that

$$\begin{aligned} p \otimes 1 + p_{(0)}\omega(p_{(1)}) &= p \otimes 1 - 1 \otimes p + 1 \otimes p + p_{(0)}\omega(p_{(1)}) \\ &= 1 \otimes p - (dp - p_{(0)}\omega(p_{(1)})) \in B \otimes P \end{aligned} \quad (2.21)$$

by property c) of  $\omega$ . The right  $H$ -colinearity of  $J_4(\omega)$  follows from property a) of  $\omega$ . The remaining needed properties of  $J_4(\omega)$  are immediate. Consequently,  $J_4$  is a mapping from  $V_4$  to  $V_1$ .

To end the proof, we need to show  $J_4 \circ J_3 \circ J_2 \circ J_1 = \text{id}$  and its three cyclicly permuted versions. We use recurrently the fact that the translation map  $\tau$  provides the inverse of the canonical map  $\chi$ , so that  $p_{(0)}p_{(1)}^{[1]} \otimes_B p_{(1)}^{[2]} = 1 \otimes_B p$  and  $h^{[1]}h^{[2]}_{(0)} \otimes h^{[2]}_{(1)} = 1 \otimes h$ .

$$\begin{aligned} (J_4 \circ J_3 \circ J_2 \circ J_1)(s)(p) &= p \otimes 1 + p_{(0)}(J_3 \circ J_2 \circ J_1)(s)(p_{(1)}) \\ &= p \otimes 1 + p_{(0)}p_{(1)}^{[1]}(J_2 \circ J_1)(s)(dp_{(1)}^{[2]}) \\ &= p \otimes 1 + (J_2 \circ J_1)(s)(dp) \\ &= p \otimes 1 + dp - J_1(s)(p) \\ &= 1 \otimes p - 1 \otimes p + s(p) = s(p) , \end{aligned} \quad (2.22)$$

$$\begin{aligned} (J_3 \circ J_2 \circ J_1 \circ J_4)(\omega)(h) &= h^{[1]}(J_2 \circ J_1 \circ J_4)(\omega)(dh^{[2]}) \\ &= h^{[1]}(d - (J_1 \circ J_4)(\omega))(h^{[2]}) \\ &= h^{[1]}(d - 1 \otimes \text{id} + J_4(\omega))(h^{[2]}) \\ &= h^{[1]}(J_4(\omega) - \text{id} \otimes 1)(h^{[2]}) \\ &= h^{[1]}h^{[2]}_{(0)}\omega(h^{[2]}_{(1)}) = \omega(h) , \end{aligned} \quad (2.23)$$

$$\begin{aligned} (J_2 \circ J_1 \circ J_4 \circ J_3)(\Pi)(dp) &= dp - (J_1 \circ J_4 \circ J_3)(\Pi)(p) \\ &= 1 \otimes p - p \otimes 1 - 1 \otimes p + (J_4 \circ J_3)(\Pi)(p) \\ &= -p \otimes 1 + p \otimes 1 + p_{(0)}(J_3(\Pi))(p_{(1)}) \\ &= p_{(0)}p_{(1)}^{[1]}\Pi(dp_{(1)}^{[2]}) = \Pi(dp) , \end{aligned} \quad (2.24)$$

$$\begin{aligned} (J_1 \circ J_4 \circ J_3 \circ J_2)(D)(p) &= 1 \otimes p - (J_4 \circ J_3 \circ J_2)(D)(p) \\ &= dp - p_{(0)}(J_3 \circ J_2)(D)(p_{(1)}) \\ &= dp - p_{(0)}p_{(1)}^{[1]}J_2(D)(dp_{(1)}^{[2]}) \\ &= dp - J_2(D)(dp) = Dp . \end{aligned} \quad (2.25)$$

This shows that the maps  $J_i$  are bijective.  $\square$

**Corollary 2.4** *If  $B \subseteq P$  is an  $H$ -Galois extension admitting a strong connection, then*

- 1)  $P$  is projective as a left  $B$ -module,
- 2)  $B$  is a direct summand of  $P$  as a left  $B$ -module,
- 3)  $P$  is left faithfully flat over  $B$ .

*Proof.* Let  $s : P \rightarrow B \otimes P$  be the splitting associated to a strong connection. Due to the unitality of  $B$  the multiplication map  $B \otimes P \rightarrow P$  is surjective. Thus  $P$  is a direct summand of  $B \otimes P$  via  $s$ , and the projectivity of  $P$  follows from the freeness of  $B \otimes P$ .

Let  $f_P$  be a unital linear functional on  $P$ . Then  $(\text{id} \otimes f_P) \circ s$  is a left  $B$  linear map splitting the inclusion  $B \subseteq P$ . Hence  $B$  is a direct summand of  $P$ .

Finally, since  $P$  is projective it is flat. On the other hand, since  $P$  contains  $B$  as a direct summand, it is also faithfully flat.  $\square$

In fact, since  $s$  embeds  $P$  in  $B \otimes P$  colinearly, we can say that  $P$  is an  $H$ -equivariantly projective left  $B$ -module. Next, we translate  $s$  to the setting of associated quantum bundles so as to be able to compute their projector matrices with the help of Lemma 1.4.

**Proposition 2.5** *Let  $s : P \rightarrow B \otimes P$  be the splitting associated to a strong connection on  $H$ -Galois extension  $B \subseteq P$ , and let  $\rho : V \rightarrow V \otimes H$  be any finite dimensional corepresentation of  $H$ . Denote by  $\bar{\ell}$  the canonical isomorphism  $B \otimes \text{Hom}(V, P) \rightarrow \text{Hom}(V, B \otimes P)$ . Then the formula*

$$s_\rho(\xi) = \bar{\ell}^{-1}(s \circ \xi) \quad (2.26)$$

*gives a left  $B$ -linear splitting of the multiplication map  $B \otimes \text{Hom}_\rho(V, P) \rightarrow \text{Hom}_\rho(V, P)$ .*

*Proof.* Note first that, since  $s$  is right colinear,  $s_\rho(\text{Hom}_\rho(V, P)) \subseteq \text{Hom}_\rho(V, B \otimes P)$ . We need to show that  $\bar{\ell}(B \otimes \text{Hom}_\rho(V, P)) = \text{Hom}_\rho(V, B \otimes P)$ , where  $\bar{\ell}(b \otimes \xi)(v) = b \otimes \xi(v)$ . For this purpose we can reason as in the proof of [HM99, Proposition 2.3] and construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \otimes \text{Hom}_\rho(V, P) & \longrightarrow & B \otimes \text{Hom}(V, P) & \xrightarrow{\text{id} \otimes \underline{\rho}} & B \otimes \text{Hom}(V, P \otimes H) \\ & & \downarrow & & \downarrow \bar{\ell} & & \downarrow \underline{\ell} \\ 0 & \longrightarrow & \text{Hom}_\rho(V, B \otimes P) & \longrightarrow & \text{Hom}(V, B \otimes P) & \xrightarrow{\bar{\rho}} & \text{Hom}(V, B \otimes P \otimes H) \end{array} \quad (2.27)$$

Here  $\underline{\rho}$  is defined by  $\underline{\rho}(\xi) = (\xi \otimes \text{id}) \circ \rho - \Delta_R \circ \xi$ , and similarly  $\bar{\rho}$ . The map  $\underline{\ell}$  is the appropriate canonical isomorphism. Completing the diagram to the left with zeroes and applying the Five Isomorphism Lemma shows that the restriction of  $\bar{\ell}$  to  $B \otimes \text{Hom}_\rho(V, P)$  is an isomorphism onto  $\text{Hom}_\rho(V, B \otimes P)$ , as needed. Thus  $s_\rho$  is a map from  $\text{Hom}_\rho(V, P)$  to  $B \otimes \text{Hom}_\rho(V, P)$ , as claimed. Explicitly,  $\bar{\ell}^{-1}$  is given by

$$\bar{\ell}^{-1}(\varphi) = \sum_i \varphi(e_i) e^i = \sum_i \varphi(e_i)^{[-1]} \otimes \varphi(e_i)^{[0]} e^i, \quad (2.28)$$

where  $\{e_i\}$  is a basis of  $V$ ,  $\{e^i\}$  its dual, and we put  $\varphi(v) = \varphi(v)^{[-1]} \otimes \varphi(v)^{[0]}$  (summation understood). Similarly, we can write  $s_\rho(\xi) = s_\rho(\xi)^{[-1]} \otimes s_\rho(\xi)^{[0]}$ . The left  $B$ -linearity of  $s_\rho$  follows from the left  $B$ -linearity of  $s$  and  $\bar{\ell}$ . Finally,  $s_\rho$  splits the multiplication map because  $s$  splits the multiplication map:

$$\begin{aligned}
(m \circ s_\rho)(\xi)(v) &= s_\rho(\xi)^{[-1]} s_\rho(\xi)^{[0]}(v) \\
&= m(\bar{\ell}(s_\rho(\xi))(v)) \\
&= m((s \circ \xi)(v)) \\
&= (m \circ s \circ \xi)(v) \\
&= \xi(v) .
\end{aligned} \tag{2.29}$$

□

Applying the standard reasoning as used in the proof of Corollary 2.4, we can infer (under the assumptions of Proposition 2.5) that  $\text{Hom}_\rho(V, P)$  is projective as a left  $B$ -module. On the other hand, if  $P$  is left faithfully flat over  $B$  and the antipode of  $H$  is bijective, one can prove that  $\text{Hom}_\rho(V, P)$  is finitely generated as a left  $B$ -module [S-P]. Thus point 3) of Corollary 2.4 leads to the following conclusion (cf. [D-M97a, Appendix B]):

**Corollary 2.6** *Let  $H$  be a Hopf algebra with a bijective antipode,  $B \subseteq P$  an  $H$ -Galois extension admitting a strong connection, and  $\rho : V \rightarrow V \otimes H$  a finite-dimensional corepresentation of  $H$ . Then the associated module of colinear maps  $\text{Hom}_\rho(V, P)$  is finitely generated projective as a left  $B$ -module.*

Closely related to  $B$ -bimodule  $\text{Hom}_\rho(V, P)$  is  $B$ -bimodule  $P_\rho := \sum_{\varphi \in \text{Hom}_\rho(V, P)} \varphi(V) \subseteq P$  (cf. [D-M97a, Appendix B]). It turns out that such submodules of  $P$  are invariant under the splitting associated to a strong connection:

**Proposition 2.7** *Let  $s$  be the splitting associated to a strong connection on an  $H$ -Galois extension  $B \subseteq P$ . Let  $\rho : V \rightarrow V \otimes H$  be a finite-dimensional corepresentation of  $H$  and  $P_\rho := \sum_{\varphi \in \text{Hom}_\rho(V, P)} \varphi(V)$ . Then  $s(P_\rho) \subseteq B \otimes P_\rho$ .*

*Proof.* If  $p \in P_\rho$  then there exists finitely many  $\tilde{\varphi}_\nu \in \text{Hom}_\rho(V, P)$  such that

$$p = \sum_\nu \tilde{\varphi}_\nu(v_\nu) = \sum_\nu \sum_{k=1}^{\dim V} v_{\nu k} \tilde{\varphi}_\nu(e_k) = \sum_{k=1}^{\dim V} \varphi_k(e_k). \tag{2.30}$$

Here  $\{e_k\}$  is a basis of  $V$  and  $\varphi_k := \sum_\nu v_{\nu k} \tilde{\varphi}_\nu$ . (Since  $v_{\nu k}$  are simply the coefficients of  $v_\nu$  with respect to  $\{e_k\}$ , we have  $\varphi_k \in \text{Hom}_\rho(V, P)$ .) Next, we can always write  $s(p) = \sum_\mu f_\mu \otimes s(p)_\mu$ , where  $\{f_\mu\}$  is a linear basis of  $B$ . (We have the strongness condition  $s(P) \subseteq B \otimes P$ .) Since  $s$  and  $\varphi_k$  are both colinear, so is their composition  $s \circ \varphi_k$ , and we have

$$\Delta_{P \otimes P}(s \circ \varphi_k)(e_\ell) = \sum_{m=1}^{\dim V} s(\varphi_k(e_m)) \otimes u_{m\ell}^\rho = \sum_{m=1}^{\dim V} \sum_\mu f_\mu \otimes (\varphi_k(e_m))_\mu \otimes u_{m\ell}^\rho, \tag{2.31}$$

where  $u_{m\ell}^\rho$  are the matrix elements of corepresentation  $\rho$ . On the other hand, remembering that  $s(P) \subseteq B \otimes P$ , we have

$$\Delta_{P \otimes P}(s \circ \varphi_k)(e_\ell) = \sum_{\mu} f_{\mu} \otimes \Delta_R(\varphi_k(e_\ell))_{\mu}. \quad (2.32)$$

Combining the above two equalities and using the linear independence of  $f_{\mu}$ , we obtain

$$\Delta_R(\varphi_k(e_\ell))_{\mu} = \sum_{m=1}^{\dim V} (\varphi_k(e_m))_{\mu} \otimes u_{m\ell}^\rho. \quad (2.33)$$

Hence we can define a bi-index family of  $\rho$ -colinear maps by the equality  $\varphi_{k\mu}(e_\ell) = (\varphi_k(e_\ell))_{\mu}$ . Consequently, due to (2.30), we have

$$s(p) = s\left(\sum_{k=1}^{\dim V} \varphi_k(e_k)\right) = \sum_{k=1}^{\dim V} \sum_{\mu} f_{\mu} \otimes (\varphi_k(e_k))_{\mu} = \sum_{\mu} f_{\mu} \otimes \left(\sum_{k=1}^{\dim V} \varphi_{k\mu}(e_k)\right) \in B \otimes P_{\rho}, \quad (2.34)$$

as claimed.  $\square$

*Remark 2.8* Just as we define  $J_1$  in the proof of Theorem 2.3, we can define the covariant derivative on  $\text{Hom}_{\rho}(V, P)$  via the formula

$$\nabla : \text{Hom}_{\rho}(V, P) \rightarrow \Omega^1 B \otimes_B \text{Hom}_{\rho}(V, P), \quad \nabla \xi = 1 \otimes \xi - s_{\rho}(\xi). \quad (2.35)$$

Using identifications in Theorem 2.3 (isomorphisms  $J_i$ ), one can check that (2.35) agrees with [HM99, (2.2)].  $\diamond$

*Remark 2.9* Assume that  $\bar{s} \in \overline{\mathcal{S}(P)}$  (see Corollary 2.2) enjoys in addition the strongness property  $\bar{s}(P) \subseteq B \otimes P$ . Then the space of connections  $\mathcal{T}(\bar{s})$  can be fully characterized as the set of all connections  $s$  satisfying

$$\exists \beta \in \Omega^1 B \quad \forall p \in P : \quad s(p) - p\beta \in B \otimes P. \quad (2.36)$$

Indeed, one can check that  $1 \otimes 1 - \bar{s}(1) \in \Omega^1 B$ , and the rest follows from the formula for  $\mathcal{T}$ . It is tempting to call such connections semi-strong.  $\diamond$

We now proceed to establishing a link between strong connections and Cuntz-Quillen connections on bimodules [CQ95, p.283]. Let  $C$  be a coalgebra and  $N_1, N_2$  right  $C$ -comodules. Denote by  $A := \text{Hom}(C, k)$  the algebra dual to  $C$ . Then  $N_1$  and  $N_2$  enjoy the following natural left  $A$ -module structure (e.g., see [M-S93, Section 1.6]):

$$A \otimes N_i \ni a \otimes n \mapsto n_{(0)}a(n_{(1)}) \in N_i, \quad i \in \{1, 2\}. \quad (2.37)$$

With respect to this structure, any  $k$ -homomorphism from  $N_1$  to  $N_2$  is right  $C$ -colinear if and only if it is right  $A^{op}$ -linear. Thus, for an  $H$ -Galois extension  $B \subseteq P$ , algebra  $P$  is a  $(B, (H^*)^{op})$ -bimodule, where  $H^* := \text{Hom}(H, k)$  is the algebra dual to  $H$  considered as a coalgebra. By Theorem 2.3 (point 2), a strong connection can be given as a right  $(H^*)^{op}$ -linear map  $D : P \rightarrow \Omega^1 B \otimes_B P$  (see (1.16)) satisfying the left Leibniz rule and vanishing on 1. Therefore it seems natural to generalize the concept of a left bimodule connection [CQ95, p.284] to

**Definition 2.10** Let  $N$  be an  $(A_1, A_2)$ -bimodule. We say that  $\nabla_L : N \rightarrow \Omega^1 A_1 \otimes_{A_1} N$  is a left bimodule connection iff it is right  $A_2$ -linear and satisfies the left Leibniz rule:  $\nabla_L(an) = a\nabla_L(n) + da \otimes_{A_1} n$ ,  $\forall a \in A_1, n \in N$ .

We can now say that a *strong connection on  $H$ -Galois extension  $B \subseteq P$*  is a left  $(B, (H^*)^{op})$ -bimodule connection on  $P$  vanishing on 1. In an analogous way, we can define a right bimodule connection  $\nabla_R$ . Then we can put them together and, in the spirit of [CQ95, p.284], define a bimodule connection as:

**Definition 2.11** Let  $N$  be an  $(A_1, A_2)$ -bimodule and  $\nabla_L$  and  $\nabla_R$  a left and right bimodule connection, respectively. We call a pair  $(\nabla_L, \nabla_R)$  a bimodule connection on  $N$ .

Reasoning precisely as in [CQ95], one can show that an  $(A_1, A_2)$ -bimodule  $N$  admits a bimodule connection if and only if it is projective as a bimodule (i.e., as a module over  $A_1 \otimes A_2^{op}$ ). In a similar fashion, one can see that strong connections correspond to equivariant connections in the algebraic-geometry setting [R-D98, (20)].

*Remark 2.12* Within the framework of the Hopf-Galois theory the right coaction  $\text{id} \otimes \Delta_R : B \otimes P \rightarrow B \otimes P \otimes H$  and the restriction  $\Delta_{B \otimes P}$  of the diagonal coaction  $\Delta_{P \otimes P}$  (1.8) coincide. Therefore one can use either of them to define the colinearity of a splitting  $s$  of the multiplication map  $B \otimes P \rightarrow P$ . In the general setting of  $C$ -Galois extensions [BH99, Definition 2.3], the diagonal coaction  $\Delta_{P \otimes P} : P \otimes P \rightarrow P \otimes P \otimes C$  (coinciding with (1.8) for Hopf-Galois extensions) can be defined by the formula  $\Delta_{P \otimes P} = (\text{id} \otimes \psi) \circ (\Delta_R \otimes \text{id})$  [BM98a, Proposition 2.2], where  $\psi : C \otimes P \rightarrow P \otimes C$  is an entwining structure and  $\Delta_R : P \rightarrow P \otimes C$ ,  $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$ , a coaction (see [BM, Section 3] for details). If  $B$  is the subalgebra of  $P$  of  $C$ -coinvariants, i.e.,  $B = \{b \in P \mid \Delta_R(bp) = b\Delta_R(p), \forall p \in P\}$ , then

$$\Delta_{P \otimes P}(b \otimes p) = (\text{id} \otimes \psi)(\Delta_R(b1) \otimes p) = (\text{id} \otimes \psi)(b1_{(0)} \otimes 1_{(1)} \otimes p) = b1_{(0)} \otimes \psi(1_{(1)} \otimes p). \quad (2.38)$$

On the other hand, if  $B \subseteq P$  is  $C$ -Galois and  $\psi$  is its canonical entwining structure [BH99, (2.5)], then, by [BH99, Theorem 2.7],  $P$  is a  $(P, C, \psi)$ -module [B-T99], so that we have  $\Delta_R(p'p) = p'_{(0)}\psi(p'_{(1)} \otimes p)$ . In particular,  $\Delta_R(p) = 1_{(0)}\psi(1_{(1)} \otimes p)$ . Hence

$$(\text{id} \otimes \Delta_R)(b \otimes p) = b \otimes 1_{(0)}\psi(1_{(1)} \otimes p). \quad (2.39)$$

Therefore we need to distinguish between  $\Delta_{B \otimes P}$  and  $\text{id} \otimes \Delta_R$  in the  $C$ -Galois case. <sup>1</sup>

If we define a strong connection on a  $C$ -Galois extension  $B \subseteq P$  as a unital left  $B$ -linear right  $C$ -colinear (with respect to  $\Delta_{B \otimes P}$ ) splitting of the multiplication map  $B \otimes P \rightarrow P$ , then such a strong connection yields a connection in the sense of [BM, Definition 3.5]. Indeed, let  $s$  be such a splitting, and  $\Pi^s(rdp) := r(s(p) - p \otimes 1)$ . One can verify that this formula gives a well-defined left  $P$ -linear endomorphism of  $\Omega^1 P$ . Furthermore, by the left  $B$ -linearity of  $s$ , for any  $\sum_i db_i.p_i \in (\Omega^1 B)P$ , we have:

$$\Pi^s\left(\sum_i db_i.p_i\right) = \sum_i \Pi^s(d(b_i.p_i) - b_i dp_i) = \sum_i s(b_i p_i) - b_i p_i \otimes 1 - b_i(s(p_i) - p_i \otimes 1) = 0. \quad (2.40)$$

---

<sup>1</sup>We are grateful to T. Brzeziński for suggesting to us this way of arguing.

Hence  $P(\Omega^1 B)P \subseteq \text{Ker } \Pi^s$  by the left  $P$ -linearity of  $\Pi^s$ . On the other hand, since  $m \circ s = \text{id}$  and  $s(P) \subseteq B \otimes P$ , we have  $\pi_B(s(p)) = 1 \otimes_B p$ , where  $\pi_B : P \otimes P \rightarrow P \otimes_B P$  is the canonical surjection. Consequently,

$$\pi_B(\Pi^s(p'dp)) = \pi_B(p'(s(p) - p \otimes 1)) = p'\pi_B(s(p)) - rp \otimes_B 1 = p' \otimes_B p - rp \otimes_B 1 = \pi_B(p'dp). \quad (2.41)$$

Therefore, since  $P(\Omega^1 B)P = \text{Ker } \pi_B$  (see below (1.23)), we obtain  $\text{Ker } \Pi^s \subseteq P(\Omega^1 B)P$ . Thus  $\text{Ker } \Pi^s = P(\Omega^1 B)P$ . Next, take any  $p \in P$ . It follows from  $s(P) \subseteq B \otimes P$  that

$$dp - \Pi^s(dp) = 1 \otimes p - p \otimes 1 - s(p) + p \otimes 1 = 1 \otimes p - s(p) \in B \otimes P. \quad (2.42)$$

Since also  $m(1 \otimes p - s(p)) = 0$ , we have  $dp - \Pi^s(dp) \in (\Omega^1 B)P \subseteq \text{Ker } \Pi^s$ . By the left  $P$ -linearity of  $\Pi^s$  we can conclude now that  $\Pi^s \circ (\text{id} - \Pi^s) = 0$ , i.e.,  $(\Pi^s)^2 = \Pi^s$ . It remains to show that  $\Delta_{P \otimes P} \circ \Pi^s \circ d = ((\Pi^s \circ d) \otimes \text{id}) \circ \Delta_R$ . The property  $\psi(c \otimes 1) = 1 \otimes c$  entails

$$\Delta_{P \otimes P}(p \otimes 1) = p_{(0)} \otimes \psi(p_{(1)} \otimes 1) = p_{(0)} \otimes 1 \otimes p_{(1)}. \quad (2.43)$$

Therefore

$$\begin{aligned} \Delta_{P \otimes P}(\Pi^s(dp)) &= \Delta_{P \otimes P}(s(p)) - \Delta_{P \otimes P}(p \otimes 1) \\ &= s(p_{(0)}) \otimes p_{(1)} - p_{(0)} \otimes 1 \otimes p_{(1)} \\ &= ((\Pi^s \circ d) \otimes \text{id})(\Delta_R(p)) \end{aligned} \quad (2.44)$$

by the colinearity of  $s$ . Consequently  $\Pi^s$  is a connection, as claimed.  $\diamond$

To exemplify Proposition 2.1 and Theorem 2.3, let us translate the strong and non-strong connection forms on quantum projective space  $\mathbb{R}P_q^2$  [H-PM96, Example 2.8] to the language of splittings.

*Example 2.13 (Quantum projective space  $\mathbb{R}P_q^2$ .)* First let us recall how to define the coordinate ring  $A(S_{q,\infty}^2)$  of the equator Podleś quantum sphere [P-P87]. To this end, we modify the convention in [H-PM96] by replacing  $q$  by  $q^{-1}$  and rewriting the generators as follows:

$$x = x_{11}, \quad y = x_{12}, \quad z = \frac{\sqrt{2(1+q^4)}}{1+q^2} x_{13}. \quad (2.45)$$

Now we can define  $A(S_{q,\infty}^2)$  as  $\mathbb{C}\langle x, y, z \rangle / I_{q,\infty}$ , where  $\mathbb{C}\langle x, y, z \rangle$  is the (unital) free algebra generated by  $x, y, z$  and  $I_{q,\infty}$  is the two-sided ideal generated by

$$x^2 + y^2 + z^2 - 1, \quad xy - yx - i \frac{q^4 - 1}{q^4 + 1} z^2, \quad xz - \frac{q^2 + q^{-2}}{2} zx - i \frac{q^{-2} - q^2}{2} zy, \quad yz - \frac{q^2 + q^{-2}}{2} zy - i \frac{q^2 - q^{-2}}{2} zx. \quad (2.46)$$

To make  $A(S_{q,\infty}^2)$  into a  $\text{Map}(\mathbb{Z}_2, \mathbb{C})$ -comodule algebra we use the formulas (see above Section 6 in [P-P87] for the related quantum-sphere automorphisms)

$$\Delta_R(x) = x \otimes \gamma, \quad \Delta_R(y) = y \otimes \gamma, \quad \Delta_R(z) = z \otimes \gamma, \quad (2.47)$$

where  $\gamma(\pm 1) = \pm 1$ . The coordinate ring of quantum projective space  $\mathbb{R}P_q^2$  is then defined as the  $\text{Map}(\mathbb{Z}_2, \mathbb{C})$ -coinvariant subalgebra of  $A(S_{q,\infty}^2)$ . (The algebra  $A(\mathbb{R}P_q^2)$  is the subalgebra of  $A(S_{q,\infty}^2)$  generated by the monomials of even degree.) The extension  $A(\mathbb{R}P_q^2) \subseteq A(S_{q,\infty}^2)$  is a



$\text{Map}(\mathbb{Z}_2, \mathbb{C})$ -Galois extension which is not cleft. The non-cleftness can be proved by reasoning exactly as in [HM99, Appendix]. Indeed, since 1 and  $\gamma$  are linearly independent group-likes ( $\Delta c = c \otimes c$ ), a cleaving map  $\Phi : \text{Map}(\mathbb{Z}_2, \mathbb{C}) \rightarrow A(S_{q,\infty}^2)$  would have to map them to linearly independent (injectivity of  $\Phi$ ) invertible (convolution invertibility of  $\Phi$ ) elements in  $A(S_{q,\infty}^2)$ . But this is impossible as  $A(S_{q,\infty}^2) \subseteq A(SL_q(2))$ , and the only invertible elements in  $A(SL_q(2))$  are non-zero numbers [HM99, Appendix]. Translating the formula [H-PM96, Proposition 2.14] for a strong connection to our setting, we have

$$\omega(\gamma) = xdx + ydy + zdz = x \otimes x + y \otimes y + z \otimes z - 1 \otimes 1. \quad (2.48)$$

The splitting corresponding to  $\omega$  is then, due to its unitality and left  $A(\mathbb{R}P_q^2)$ -linearity, determined by

$$s(x) = xt, \quad s(y) = yt, \quad s(z) = zt, \quad \text{where} \quad t = x \otimes x + y \otimes y + z \otimes z. \quad (2.49)$$

Thus one can directly see that the image of  $s$  is in  $A(\mathbb{R}P_q^2) \otimes A(S_{q,\infty}^2)$ .

Next, consider a non-strong connection  $\tilde{\omega}(\gamma) = \omega(\gamma) - 2dx^2$  [H-PM96, Proposition 2.15]. Again, we compute the corresponding splitting:

$$\tilde{s}(x) = s(x) - 2xdx^2, \quad \tilde{s}(y) = s(y) - 2ydx^2, \quad \tilde{s}(z) = s(z) - 2zdx^2. \quad (2.50)$$

As in the proof of [H-PM96, Proposition 2.15], we can invoke the representation theory contained in [P-P87] to conclude that  $xdx^2 \neq 0$ . Consequently,

$$((\text{id} \otimes \text{flip}) \circ (\Delta_R \otimes \text{id} - \text{id} \otimes \text{id} \otimes \text{id}))(xdx^2) = xdx^2 \otimes (\gamma - 1) \neq 0. \quad (2.51)$$

Hence the image of  $\tilde{s}$  is not in  $A(\mathbb{R}P_q^2) \otimes A(S_{q,\infty}^2)$ .  $\diamond$

*Remark 2.14* Let  $x, y, z$  be as above. Since  $x^2 + y^2 + z^2 = 1$  (which was the reason for rescaling the generators) and, with respect to the star structure inherited from  $SU_q(2)$ , we have  $x^* = x$ ,  $y^* = y$ ,  $z^* = z$ , we can treat the generators  $x, y, z$  as the Cartesian coordinates of  $S_{q,\infty}^2$ . Having this in mind, we take the idempotent  $F = (x, y, z)^T(x, y, z) \in M_3(A(S_{q,\infty}^2))$  (here  $^T$  stands for the matrix transpose) and define the projective module of the normal bundle of  $S_{q,\infty}^2$  as  $A(S_{q,\infty}^2)^3 F$ . Therefore one can define the projective module of the *tangent bundle of the equator Podleś quantum sphere* as  $A(S_{q,\infty}^2)^3(I_3 - F)$ , where  $I_3$  is the identity matrix in  $M_3(A(S_{q,\infty}^2))$ .  $\diamond$

Let us now consider strong connections on *principal homogeneous Hopf-Galois extensions*, i.e.,  $P/I$ -Galois extensions given by a Hopf ideal  $I$  in a Hopf algebra  $P$ . Here the coaction is given by the formula  $\Delta_R = (\text{id} \otimes \pi_I) \circ \Delta$ , where  $\pi_I$  is the canonical surjection  $P \rightarrow P/I$ . For such extensions, it is known (e.g., see [DHS99, Theorem 2.1]) that if  $B = P^{co P/I}$  then  $I = B^+P$ , where  $B^+ = \text{Ker } \varepsilon \cap B$ . If  $s$  is the splitting associated to a strong connection, then, due to the left  $B$ -linearity of  $s$ ,

$$s(B^+P) = B^+s(P) \subseteq B^+B \otimes P = B^+ \otimes P. \quad (2.52)$$

Hence  $s$  descends to a splitting  $i$  of the canonical surjection  $P \rightarrow P/(B^+P)$ :

$$\begin{array}{ccc}
P & \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{m} \end{array} & B \otimes P \\
\downarrow & & \downarrow \\
H & \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{\quad} \end{array} & (B \otimes P)/(B^+ \otimes P) = P.
\end{array} \tag{2.53}$$

Explicitly, we have  $i(\bar{p}) = ((\varepsilon \otimes \text{id}) \circ s)(p)$ . (The map is well-defined because of (2.52).) Put  $s(p) = s(p)^{[0]} \otimes s(p)^{[1]}$  (summation understood). Then, as  $m \circ s = \text{id}$  and, for  $b \in B$ ,  $p \in P$ ,  $\varepsilon(b)p = bp \bmod B^+P$ , we have

$$(\pi_I \circ i)(\bar{p}) = \pi_I(\varepsilon(s(p)^{[0]})s(p)^{[1]}) = \pi_I(s(p)^{[0]}s(p)^{[1]}) = (\pi_I \circ m \circ s)(p) = \bar{p}. \tag{2.54}$$

Furthermore, since  $s$  is unital, so is  $i$ . The right colinearity of  $i$  follows from the strongness  $(s(P) \subseteq B \otimes P)$  and the right colinearity of  $s$ :

$$\begin{aligned}
(\Delta_R \circ i)(\bar{p}) &= \varepsilon(s(p)^{[0]})s(p)^{[1]}_{(0)} \otimes s(p)^{[1]}_{(1)} \\
&= ((\varepsilon \otimes \text{id} \otimes \text{id}) \circ \Delta_{P \otimes P} \circ s)(p) \\
&= ((\varepsilon \otimes \text{id}) \circ s)(p_{(1)}) \otimes \overline{p_{(2)}} \\
&= i(\overline{p_{(1)}}) \otimes \overline{p_{(2)}} = i(\bar{p}_{(1)}) \otimes \bar{p}_{(2)}.
\end{aligned} \tag{2.55}$$

Thus one can associate to any strong connection on a principal homogeneous Hopf-Galois extension a total integral of Doi [D-Y85] (unital right colinear map  $H \rightarrow P$ ). Recall that total integrals always exist on faithfully flat Hopf-Galois extensions ([S-HJ90a, Theorem 1], [D-Y85, (1.6)], [S-HJ90a, Remark 3.3]). This is in agreement with point 3 of Corollary 2.4, although we claim there only the left faithful flatness, and faithfully flat Hopf-Galois extensions  $B \subseteq P$  are defined as Hopf-Galois extensions such that  $P$  is  $B$ -faithfully-flat on both sides. Note also that we could equally well proceed as in [BM98b, Proposition 3.6] and define  $i$  via a connection form. If  $s$  is the splitting associated to a connection form  $\omega$ , i.e.,  $s = J_4(\omega)$  (see (2.20)), then

$$\begin{aligned}
i(\bar{p}) &= ((\varepsilon \otimes \text{id}) \circ J_4(\omega))(p) \\
&= (\varepsilon \otimes \text{id})(p \otimes 1 + p_{(1)}\omega(\overline{p_{(2)}})) \\
&= \varepsilon(p) \otimes 1 + \varepsilon(p_{(1)})\varepsilon(\omega(\overline{p_{(2)}})^{(1)}) \otimes \omega(\overline{p_{(2)}})^{(2)} \\
&= \varepsilon_H(\bar{p}) \otimes 1 + \varepsilon(p_{(1)})(\varepsilon \otimes \text{id})(\omega(\overline{p_{(2)}})) \\
&= \varepsilon_H(\bar{p}) \otimes 1 + ((\varepsilon \otimes \text{id}) \circ \omega)(\bar{p}),
\end{aligned} \tag{2.56}$$

where  $\omega(h) = \omega(h)^{(1)} \otimes \omega(h)^{(2)}$ , summation understood, and  $\varepsilon_H$  denotes the counit on  $H$ . (See [BM98b, Proposition 3.6] for this kind of splittings in the case of non-universal calculus.) If  $i$  is also left colinear, then, by [HM99, Proposition 2.4], the formula  $\omega = (S * \text{d}) \circ i$  associates to  $i$  a strong connection. (Such connections are called canonical strong connections.) It turns out that applying the above described way of associating a total integral to a strong connection in the canonical case is simply solving the equation  $\omega = (S * \text{d}) \circ i$  for  $i$ . Indeed, since  $\omega(h) = Si(h)_{(1)}di(h)_{(2)}$ , we have

$$J_4(\omega)(p) = p \otimes 1 + p_{(1)}\omega(\overline{p_{(2)}}) = p_{(1)}Si(\overline{p_{(2)}})_{(1)} \otimes i(\overline{p_{(2)}})_{(2)}. \tag{2.57}$$

Applying  $\varepsilon \otimes \text{id}$  yields

$$i(\bar{p}) = ((\varepsilon \otimes \text{id}) \circ J_4(\omega))(p), \tag{2.58}$$

as claimed.

*Example 2.15 (Quantum and classical Hopf fibration.)* The above described formalism applies to the quantum Hopf fibration. We refer to [HM99] for the computation of projector matrices of the quantum Hopf line bundles from the Dirac q-monopole connection [BM93], and to [H-PM] for the computation of the Chern-Connes pairing of these matrices with the cyclic cocycle (trace) [MNW91, (4.4)]. (This pairing yields numbers called “Chern numbers” or “charges.” See Proposition 3.7 for the freeness of the direct sum of charge  $-1$  and charge  $1$  quantum Hopf line bundles, cf. [DS94, (4.2)] for a local description of such bundles.) Here we only remark that this quantum principal fibration admits infinitely many canonical strong connections. Indeed, for the injective antipode (which is the case here), [HM99, Corrolary 2.6] classifies the canonical strong connections by unital bilinear splittings. On the other hand, by [MMNNU91, p.363], all unital bilinear splittings  $i : \mathbb{C}[z, z^{-1}] \rightarrow A(SL_q(2))$  are of the form:

$$\begin{aligned} i(z^n) &= (1 + \zeta p_n(\zeta))\alpha^n, \\ i(z^{-n}) &= (1 + \zeta r_n(\zeta))\delta^n, \end{aligned} \quad (2.59)$$

where  $p_n, r_n$  are arbitrary polynomials in  $\zeta := -q^{-1}\beta\gamma$ . (Here,  $\alpha, \beta, \gamma, \delta$  are the generators of  $A(SL_q(2))$  as in [HM99].) Since the equality  $\omega = (S * d) \circ i$  can be solved for  $i$  (see (2.58)), different splittings yield different connections. Hence there are infinitely many connections.

However, for  $q = 1$ , after passing to the de Rham forms, all the canonical strong connections coincide with the classical Dirac monopole. More precisely, let  $\pi_{DR}$  be the canonical projection from the universal onto the de Rham differential calculus and  $i_0$  be the splitting corresponding to the Dirac monopole (i.e., given by (2.59) with  $p_n = 0 = r_n$  for all  $n$ ). Then

$$\pi_{DR} \circ (S * d) \circ i = \pi_{DR} \circ (S * d) \circ i_0 \quad \text{for all } i. \quad (2.60)$$

Indeed, we have

$$((S * d_{DR}) \circ (i - i_0))(z) = (S * d_{DR})(\beta\gamma p_1(\beta\gamma)\alpha). \quad (2.61)$$

Furthermore, using the commutativity of functions with forms and functions, and the Leibniz rule, we obtain

$$\begin{aligned} (S * d_{DR})(hh') &= S(h_{(1)})S(h'_{(1)})d_{DR}(h_{(2)}h'_{(2)}) \\ &= S(h'_{(1)})h'_{(2)}S(h_{(1)})d_{DR}h_{(2)} + S(h_{(1)})h_{(2)}S(h'_{(1)})d_{DR}h'_{(2)} \\ &= \varepsilon(h')S(h_{(1)})d_{DR}h_{(2)} + \varepsilon(h)S(h'_{(1)})d_{DR}h'_{(2)}. \end{aligned} \quad (2.62)$$

Substituting  $h = \beta$  and  $h' = \gamma p_1(\beta\gamma)\alpha$ , and noting that  $\varepsilon(\beta) = 0$  and  $\varepsilon(\beta\gamma p_1(\beta\gamma)\alpha) = 0$ , one can conclude that  $(S * d_{DR})(i(z)) = (S * d_{DR})(i_0(z))$ . On the other hand, for any connection form  $\omega$  we have

$$(\pi_{DR} \circ \omega)(uu') = (\pi_{DR} \circ \omega)(u)\varepsilon(u') + \varepsilon(u)(\pi_{DR} \circ \omega)(u'). \quad (2.63)$$

Therefore  $(S * d_{DR}) \circ i$  and  $(S * d_{DR}) \circ i_0$  coincide on any power of  $z$ , whence are equal, as claimed.  $\diamond$

### 3 Chern-Connes pairing for the super Hopf fibration

The super Hopf fibration leading to the super sphere has an interesting history. To the best of our knowledge, it was first introduced by Landi and Marmo [LM87]. They treated supersymmetric abelian gauge fields in general and worked out details for the super group  $UOSP(1,2)$ .

Everything was formulated within the Grassmann envelope of the super algebra  $uosp(1,2)$ . The super manifold theory has been used in the work of Teofilatto [T-P88]. He defines and studies super Riemann surfaces. As the simplest example, he treated the super sphere with  $S^2$  as its body. Ideas of noncommutative geometry were used in [GKP96, GKP97] to introduce an ultraviolet regularization for quantum fields defined on  $S^2$ . The fuzzy sphere [M-J92] was introduced through suitable embeddings of the algebra of  $N \times N$  matrices. In [GKP96], similar embeddings of modules led to approximation of sections of line bundles over  $S^2$ . Also in [GKP96], there is a study of fermions and supersymmetric extensions of the fuzzy sphere. An extensive treatment of the approximation of super-graded functions over the super sphere, and sections of a bundle through sequences of graded modules, as well as the treatment of the graded de Rham complex, is given in [GR98]. The description of the monopole on the super sphere that we provide can be related to that given in [BBL90]. A detailed study of the super monopole using the super-geometry approach can be found in [L-Gb].

Our approach here to the super Hopf fibration is purely algebraic. First, we show that the super Hopf fibration can be considered as an  $H$ -Galois extension  $A(S_s^2) \subseteq A(S_s^3)$ , where  $H = \mathbb{C}[z, z^{-1}]$  is the Hopf algebra generated by invertible group-like element  $z$ . The polynomial algebras  $A(S_s^3)$  and  $A(S_s^2)$  are taken as nilpotent extensions (by two Grassmann variables  $\lambda_\pm$ ) of the (complex) coordinate rings of 3-dimensional sphere  $S^3$  and 2-dimensional sphere  $S^2$ , respectively (see [GKP96]). This is summed up in the following commutative diagram with exact columns (but not rows):

$$\begin{array}{ccccc}
A(S_s^2) \cap \langle \lambda_\pm \rangle & \longrightarrow & \langle \lambda_\pm \rangle & & \\
\downarrow & & \downarrow & & \\
A(S_s^2) & \longrightarrow & A(S_s^3) & \longrightarrow & H \\
\downarrow \varphi & & \downarrow & & \parallel \\
A(S^2) & \longrightarrow & A(S^3) & \longrightarrow & H
\end{array} \tag{3.1}$$

Thus, in a sense, the super Hopf fibration can be viewed as a Grassmann covering of the classical (complex) Hopf fibration.

**Definition 3.1** *Let  $R = \mathbb{C}[a, b, c, d]$  be the polynomial ring in four variables. Put  $D = ad - bc$ . Let  $I$  be the two-sided ideal in the (unital) free algebra  $R\langle \lambda_+, \lambda_- \rangle$  generated by*

$$\lambda_+^2, \quad \lambda_-^2, \quad \lambda_+ \lambda_- + \lambda_- \lambda_+, \quad \lambda_+ \lambda_- + D - 1. \tag{3.2}$$

*We call the quotient algebra  $A(S_s^3) := R\langle \lambda_+, \lambda_- \rangle / I$  the coordinate ring of 3-dimensional super sphere  $S_s^3$ .*

It can be easily verified that the (matrix) formula

$$\Delta_R \begin{pmatrix} a & b \\ c & d \\ \lambda_+ & \lambda_- \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \\ \lambda_+ \otimes 1 & \lambda_- \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes z & 0 \\ 0 & 1 \otimes z^{-1} \end{pmatrix} \tag{3.3}$$

defines a coaction  $\Delta_R : A(S_s^3) \rightarrow A(S_s^3) \otimes H$  making  $A(S_s^3)$  a right  $H$ -comodule algebra.

**Lemma 3.2** *Let  $A(S_s^2) := \{a \in A(S_s^3) \mid \Delta_R(a) = a \otimes 1\}$  be the algebra of  $H$ -coinvariants. Then  $A(S_s^2)$  is the subalgebra of  $A(S_s^3)$  generated by*

$$1, \quad ab, \quad bc, \quad cd, \quad \lambda_+b, \quad \lambda_+d, \quad \lambda_-a, \quad \lambda_-c, \quad \lambda_+\lambda_-.$$
 (3.4)

*Proof.* Evidently, the algebra generated by (3.4) is contained in  $A(S_s^2)$ . For the opposite inclusion, note first that every non-zero element  $a$  of  $A(S_s^3)$  can be written as a linear combination of non-zero monomials  $m_{k,\ell}$  such that  $\Delta_R(m_{k,\ell}) = m_{k,\ell} \otimes z^k$ . Since the powers of  $z$  form a basis of  $H$ , if  $a \in A(S_s^2)$ , then  $a$  must be a linear combination of non-zero monomials  $m_{0,\ell}$ . On the other hand, any  $m_{0,\ell}$  is a word composed of the same number of letters coming from the alphabet  $\{a, c, \lambda_+\}$  and the alphabet  $\{b, d, \lambda_-\}$ . Furthermore, since all letters commute or anti-commute, we can always pair the letters coming from different alphabets. Hence  $m_{0,\ell}$  can be expressed in terms of (3.4), as needed.  $\square$

**Proposition 3.3** *The extension of algebras  $A(S_s^2) \subseteq A(S_s^3)$  is  $H$ -Galois.*

*Proof.* Define the map  $\tau : H \rightarrow A(S_s^3) \otimes_{A(S_s^2)} A(S_s^3)$  by the formulas ( $n \in \mathbb{N}$ ):

$$\begin{aligned} \tau(z^n) &= (1 + n\lambda_+\lambda_-) \sum_{k=0}^n \binom{n}{k} d^{n-k} (-b)^k \otimes_{A(S_s^2)} a^{n-k} c^k, \\ \tau(z^{-n}) &= (1 + n\lambda_+\lambda_-) \sum_{k=0}^n \binom{n}{k} a^{n-k} (-c)^k \otimes_{A(S_s^2)} d^{n-k} b^k. \end{aligned}$$
 (3.5)

We are going to prove that  $\tilde{\chi} := (m \otimes \text{id}) \circ (\text{id} \otimes \tau)$  is the inverse of the canonical map  $\chi$ . (This means that  $\tau$  is the translation map.) Since  $\chi$  and  $\tilde{\chi}$  are both left  $A(S_s^3)$ -linear maps by construction, it suffices to check  $\chi \circ \tilde{\chi} = \text{id}$  and  $\tilde{\chi} \circ \chi = \text{id}$  on elements of the form  $1 \otimes h$  and  $1 \otimes_{A(S_s^2)} p$ , respectively. To verify the first identity, we recall that any  $h \in H$  is a linear combination of  $z^{\pm n}$ ,  $n \in \mathbb{N}$ , and compute:

$$\begin{aligned} (\chi \circ \tilde{\chi})(1 \otimes z^n) &= (1 + n\lambda_+\lambda_-) \sum_{k=0}^n \binom{n}{k} d^{n-k} (-b)^k \chi(1 \otimes_{A(S_s^2)} a^{n-k} c^k) \\ &= (1 + n\lambda_+\lambda_-) \left( \sum_{k=0}^n \binom{n}{k} d^{n-k} (-b)^k a^{n-k} c^k \right) \otimes z^n \\ &= (1 + n\lambda_+\lambda_-) (ad - bc)^n \otimes z^n \\ &= (1 + n\lambda_+\lambda_-) (1 - \lambda_+\lambda_-)^n \otimes z^n \\ &= (1 + n\lambda_+\lambda_-) (1 - n\lambda_+\lambda_-) \otimes z^n \\ &= 1 \otimes z^n. \end{aligned}$$
 (3.6)

In the fourth equality we used the determinant relation  $\lambda_+\lambda_- + ad - bc = 1$ . Similarly, we obtain:

$$(\chi \circ \tilde{\chi})(1 \otimes z^{-n}) = (1 + n\lambda_+\lambda_-) \sum_{k=0}^n \binom{n}{k} a^{n-k} (-c)^k \chi(1 \otimes_{A(S_s^2)} d^{n-k} b^k) = 1 \otimes z^{-n}. \quad (3.7)$$

Thus  $\chi \circ \tilde{\chi} = \text{id}$ . For the other identity, we note that it is sufficient to check it on the monomials  $m_{\pm n}$  (as in the proof of Lemma 3.2 but with the second index suppressed). Since

$\Delta_R(m_{\pm n}) = m_{\pm n} \otimes z^{\pm n}$ , we have  $m_n d^{n-k} b^k \in A(S_s^2)$  and  $m_{-n} a^{n-k} c^k \in A(S_s^2)$ . Hence, using the centrality of  $\lambda_+ \lambda_- \in A(S_s^2)$ , we can compute:

$$\begin{aligned}
(\tilde{\chi} \circ \chi)(1 \otimes_{A(S_s^2)} m_n) &= m_n \tilde{\chi}(1 \otimes z^n) \\
&= m_n (1 + n\lambda_+ \lambda_-) \sum_{k=0}^n \binom{n}{k} d^{n-k} (-b)^k \otimes_{A(S_s^2)} a^{n-k} c^k \\
&= (1 + n\lambda_+ \lambda_-) \sum_{k=0}^n \binom{n}{k} m_n d^{n-k} (-b)^k \otimes_{A(S_s^2)} a^{n-k} c^k \\
&= 1 \otimes_{A(S_s^2)} (1 + n\lambda_+ \lambda_-) \sum_{k=0}^n \binom{n}{k} m_n d^{n-k} (-b)^k a^{n-k} c^k \\
&= 1 \otimes_{A(S_s^2)} m_n (1 + n\lambda_+ \lambda_-) (ad - bc)^n \\
&= 1 \otimes_{A(S_s^2)} m_n.
\end{aligned} \tag{3.8}$$

Here the last step is as in the previous calculation. Similarly, we get:

$$\begin{aligned}
(\tilde{\chi} \circ \chi)(1 \otimes_{A(S_s^2)} m_{-n}) &= (1 + n\lambda_+ \lambda_-) \sum_{k=0}^n \binom{n}{k} m_{-n} a^{n-k} (-c)^k \otimes_{A(S_s^2)} d^{n-k} b^k \\
&= 1 \otimes_{A(S_s^2)} m_{-n}.
\end{aligned} \tag{3.9}$$

Therefore  $\tilde{\chi}$  is the inverse of  $\chi$ , and the extension is  $H$ -Galois.  $\square$

We use the idea of colinear lifting (1.25) to construct a connection form. We consider this connection as the (universal-calculus) super Dirac monopole. Since it is strong, we can conclude that the extension  $A(S_s^2) \subseteq A(S_s^3)$  enjoys all properties itemized in Corollary 2.4.

**Proposition 3.4** *Let  $\omega : H \rightarrow \Omega^1 A(S_s^3)$  be the linear map defined by ( $n \in \mathbb{N}$ )*

$$\omega(z^n) = (1 + n\lambda_+ \lambda_-) \sum_{k=0}^n \binom{n}{k} d^{n-k} (-b)^k d(a^{n-k} c^k), \tag{3.10}$$

$$\omega(z^{-n}) = (1 + n\lambda_+ \lambda_-) \sum_{k=0}^n \binom{n}{k} a^{n-k} (-c)^k d(d^{n-k} b^k). \tag{3.11}$$

*Then  $\omega$  is a strong connection form.*

*Proof.* Note first that  $\omega(1) = 0$  and  $\Delta_{\Omega^1 P} \omega(z^{\pm n}) = \omega(z^{\pm n}) \otimes 1$ . Furthermore,

$$((m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) \circ \omega)(z^{\pm n}) = \chi(\tau(z^{\pm n}) - 1 \otimes_{A(S_s^2)} 1) = 1 \otimes (z^{\pm n} - \varepsilon(z^{\pm n})). \tag{3.12}$$

This proves that  $\omega$  is a connection form. It remains to check the strongness condition. By linearity, it suffices to do it on monomials  $m_{\pm n}$  (see the proof of Proposition 3.3). Putting

$$\begin{aligned}
p_k^+ &= (1 + n\lambda_+ \lambda_-) \binom{n}{k} d^{n-k} (-b)^k, & q_k^+ &= a^{n-k} c^k, \\
p_k^- &= (1 + n\lambda_+ \lambda_-) \binom{n}{k} a^{n-k} (-c)^k, & q_k^- &= d^{n-k} b^k,
\end{aligned} \tag{3.13}$$

and using the Leibniz rule, we obtain:

$$\begin{aligned}
&dm_{\pm n} - m_{\pm n} \omega(z^{\pm n}) \\
&= dm_{\pm n} - \sum_{k=0}^n m_{\pm n} p_k^{\pm} dq_k^{\pm} \\
&= dm_{\pm n} - d(m_{\pm n} \sum_{k=0}^n p_k^{\pm} q_k^{\pm}) + \sum_{k=0}^n d(m_{\pm n} p_k^{\pm}) \cdot q_k^{\pm} \\
&= \sum_{k=0}^n d(m_{\pm n} p_k^{\pm}) \cdot q_k^{\pm} \in (\Omega^1 A(S_s^2)) A(S_s^3).
\end{aligned} \tag{3.14}$$

Consequently, for any  $a \in A(S_s^3)$ , we have  $(\text{id} - \Pi^\omega)(da) \in (\Omega^1 A(S_s^2))A(S_s^3)$ , i.e.,  $\omega$  is strong.  $\square$

Our next step is to consider super Hopf line bundles. Precisely as in the case of quantum Hopf line bundles [HM99, Definition 3.1], since we are dealing with one-dimensional corepresentations of  $H$  ( $\rho_\mu(1) = 1 \otimes z^{-\mu}$ ), we can identify the colinear maps  $\xi : \mathbb{C} \rightarrow A(S_s^3)$  with their values at 1 ( $\eta(\xi) := \xi(1)$ ), and define them as the following bimodules over  $A(S_s^2)$ :

$$A(S_s^3)_\mu := \{a \in A(S_s^3) \mid \Delta_R a = a \otimes z^{-\mu}\}, \quad \mu \in \mathbb{Z}. \quad (3.15)$$

Reasoning in a similar manner as in the proof of Lemma 3.2, one can see that ( $n > 0$ )

$$A(S_s^3)_{-n} = \sum_{k=0}^n A(S_s^2) a^{n-k} c^k + \sum_{k=0}^{n-1} A(S_s^2) a^{n-1-k} c^k \lambda_+, \quad (3.16)$$

$$A(S_s^3)_n = \sum_{k=0}^n A(S_s^2) d^{n-k} b^k + \sum_{k=0}^{n-1} A(S_s^2) d^{n-1-k} b^k \lambda_-. \quad (3.17)$$

Note that, since the powers of  $z$  form a basis of  $H$ , we have the direct sum decomposition  $A(S_s^3) = \bigoplus_{\mu \in \mathbb{Z}} A(S_s^3)_\mu$  as  $A(S_s^2)$ -bimodules. Observe also that the bimodules  $A(S_s^3)_\mu$  provide examples of bimodules  $P_\rho$  defined in Proposition 2.7 (cf. [D-M97a, Appendix B]). Our goal is to compute projector matrices of these modules and their pairing with the appropriate cyclic cocycle on  $A(S_s^2)$ . The strategy for computing the projector matrices is to use the splitting associated to super Dirac monopole (Proposition 3.4) and Lemma 1.4. To apply the aforementioned lemma, first we need to show that the monomials occurring in formula (3.16) are linearly independent, and that the same holds for the monomials in (3.17).

### Lemma 3.5

$$\sum_{k=0}^n \alpha_k a^{n-k} c^k + \sum_{\ell=0}^{n-1} \beta_\ell a^{n-1-\ell} c^\ell \lambda_+ = 0 \quad \Rightarrow \quad \alpha_k = 0 = \beta_\ell, \quad \forall k, \ell; \quad (3.18)$$

$$\sum_{k=0}^n \alpha_k d^{n-k} b^k + \sum_{\ell=0}^{n-1} \beta_\ell d^{n-1-\ell} b^\ell \lambda_- = 0 \quad \Rightarrow \quad \alpha_k = 0 = \beta_\ell, \quad \forall k, \ell. \quad (3.19)$$

*Proof.* Let  $\tilde{R} = \mathbb{C}[\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}] / \langle \tilde{a}\tilde{d} - \tilde{b}\tilde{c} - 1 \rangle$  denote the coordinate ring of  $SL(2, \mathbb{C})$  and  $\mathbb{C}[\lambda] / \langle \lambda^2 \rangle$  be the algebra of dual numbers. We have the following homomorphism of algebras:

$$\begin{aligned} \pi : A(S_s^3) &\longrightarrow \tilde{R} \otimes \mathbb{C}[\lambda] / \langle \lambda^2 \rangle, \\ \pi(a) &= \tilde{a} \otimes 1, \quad \pi(b) = \tilde{b} \otimes 1, \quad \pi(c) = \tilde{c} \otimes 1, \quad \pi(d) = \tilde{d} \otimes 1, \quad \pi(\lambda_\pm) = 1 \otimes \lambda. \end{aligned} \quad (3.20)$$

Applying  $\pi$  to the first equality in (3.18) yields

$$\sum_{k=0}^n \alpha_k \tilde{a}^{n-k} \tilde{c}^k \otimes 1 + \sum_{\ell=0}^{n-1} \beta_\ell \tilde{a}^{n-1-\ell} \tilde{c}^\ell \otimes \lambda = 0. \quad (3.21)$$

Since the monomials  $\tilde{a}^{n-k} \tilde{c}^k$  are part of the PBW basis of  $\tilde{R}$ , they are linearly independent. Hence  $\alpha_k = 0 = \beta_\ell, \forall k, \ell$ , by the linear independence of 1 and  $\lambda$ . The second implication can be proved in the same way.  $\square$

Note now that the above described identification  $\eta$  allows one to identify  $s_{\rho_\mu}$  of (2.26) with the restriction of  $s$  to  $A(S_s^3)_\mu$  (see Proposition 2.7):

$$A(S_s^3)_\mu \ni \tilde{\xi} \mapsto ((\text{id} \otimes \eta) \circ s_{\rho_\mu} \circ \eta^{-1})(\tilde{\xi}) \in A(S_s^2) \otimes A(S_s^3)_\mu, \quad (3.22)$$

$$((\text{id} \otimes \eta) \circ s_{\rho_\mu} \circ \eta^{-1})(\tilde{\xi}) = \bar{\ell}(s_{\rho_\mu}(\eta^{-1}(\tilde{\xi}))) (1) = (s \circ \eta^{-1}(\tilde{\xi}))(1) = s(\tilde{\xi}). \quad (3.23)$$

On the other hand, remembering the formula for the universal differential and using again the fact that  $(ad - bc)^n = 1 - n\lambda_+\lambda_-$ , we can write (3.10) in the following form:

$$\omega(z^n) = (1 + n\lambda_+\lambda_-) \sum_{\ell=0}^n \binom{n}{\ell} d^{n-\ell}(-b)^\ell \otimes a^{n-\ell}c^\ell - 1 \otimes 1. \quad (3.24)$$

Substituting this to (2.20), we obtain

$$\begin{aligned} s(a^{n-k}c^k) &= a^{n-k}c^k \otimes 1 + a^{n-k}c^k \omega(z^n) \\ &= \sum_{\ell=0}^n a^{n-k}c^k (1 + n\lambda_+\lambda_-) \binom{n}{\ell} d^{n-\ell}(-b)^\ell \otimes a^{n-\ell}c^\ell, \\ s(a^{n-1-k}c^k\lambda_+) &= a^{n-1-k}c^k\lambda_+ \otimes 1 + a^{n-1-k}c^k\lambda_+ \omega(z^n) \\ &= \sum_{\ell=0}^n a^{n-1-k}c^k\lambda_+ \binom{n}{\ell} d^{n-\ell}(-b)^\ell \otimes a^{n-\ell}c^\ell. \end{aligned} \quad (3.25)$$

Similarly, substituting

$$\omega(z^{-n}) = (1 + n\lambda_+\lambda_-) \sum_{\ell=0}^n \binom{n}{\ell} a^{n-\ell}(-c)^\ell \otimes d^{n-\ell}b^\ell - 1 \otimes 1 \quad (3.26)$$

to (2.20), we get

$$\begin{aligned} s(d^{n-k}b^k) &= \sum_{\ell=0}^n d^{n-k}b^k (1 + n\lambda_+\lambda_-) \binom{n}{\ell} a^{n-\ell}(-c)^\ell \otimes d^{n-\ell}b^\ell, \\ s(d^{n-1-k}b^k\lambda_-) &= \sum_{\ell=0}^n d^{n-1-k}b^k\lambda_- \binom{n}{\ell} a^{n-\ell}(-c)^\ell \otimes d^{n-\ell}b^\ell. \end{aligned} \quad (3.27)$$

Hence, by Lemma 3.5 and Lemma 1.4, we can conclude that  $A(S_s^3)_{-n} = A(S_s^2)^{2n+1}E_{-n}$  as left  $A(S_s^2)$ -modules, where  $E_{-n} = P_{-n}Q_{-n}^T$  (symbol  $T$  stands for the matrix transpose) with

$$\begin{aligned} P_{-n}^T &:= (1 + n\lambda_+\lambda_-)(a^n, \dots, a^{n-k}c^k, \dots, c^n, a^{n-1}\lambda_+, \dots, a^{n-1-k}c^k\lambda_+, \dots, c^{n-1}\lambda_+), \\ Q_{-n}^T &:= (d^n, \dots, \binom{n}{\ell}d^{n-\ell}(-b)^\ell, \dots, (-b)^n, 0, \dots, 0). \end{aligned} \quad (3.28)$$

In an analogous manner, we infer that  $A(S_s^3)_n = A(S_s^2)^{2n+1}E_n$  as left  $A(S_s^2)$ -modules, where  $E_n = P_nQ_n^T$  with

$$\begin{aligned} P_n^T &:= (1 + n\lambda_+\lambda_-)(d^n, \dots, d^{n-k}b^k, \dots, b^n, d^{n-1}\lambda_-, \dots, d^{n-1-k}b^k\lambda_-, \dots, b^{n-1}\lambda_-), \\ Q_n^T &:= (a^n, \dots, \binom{n}{\ell}a^{n-\ell}(-c)^\ell, \dots, (-c)^n, 0, \dots, 0). \end{aligned} \quad (3.29)$$

To show the non-freeness of the above projective modules, we determine the Chern-Connes pairing between their classes in  $K_0(A(S_s^2))$  and the cyclic cocycle on  $A(S_s^2)$  obtained by the pull-back  $\wp^*$  (see (3.1)) of the cyclic 2-cocycle  $c_2$  on  $A(S^2)$  given by the integration on  $S^2$ . We have:

$$\langle \wp^*(c_2), [E_{\pm n}] \rangle = \langle c_2, [\wp_* E_{\pm n}] \rangle = \pm n. \quad (3.30)$$

Here the last equality follows from the fact that matrix  $(\wp_* E_{\pm n})_{i,j} := \wp((E_{\pm n})_{i,j})$  is a projector matrix of the classical Hopf line bundle with the Chern number equal to  $\pm n$ . Furthermore, since every free module can be represented in  $K_0$  by the identity matrix, the  $K_0$ -class of any free  $A(S_s^2)$ -module always vanishes. (The Chern class of a trivial bundle is zero.) Thus the left modules  $A(S_s^3)_\mu$ ,  $\mu \neq 0$ , are not (stably) free and are pairwise non-isomorphic. Now, reasoning as in [HM99, Section 4], we obtain:



**Corollary 3.6** *The  $H$ -Galois extension  $A(S_s^2) \subseteq A(S_s^3)$  (super Hopf fibration) is not cleft.*

Let us remark that projectors  $E_{\pm n}$  are not hermitian with respect to the involution

$$a^* = d, \quad b^* = -c, \quad c^* = -b, \quad d^* = a, \quad \lambda_{\pm}^* = -\lambda_{\mp}. \quad (3.31)$$

Nevertheless, one can slightly modify  $E_{\pm n}$  to find hermitian projectors  $F_{\pm n} = F_{\pm n}^\dagger$  such that the modules  $A(S_s^2)^{2n+1}E_{\pm n}$  and  $A(S_s^2)^{2n+1}F_{\pm n}$  are isomorphic. They are given by the formulas  $F_{\pm n} = U_{\pm n}U_{\pm n}^\dagger$ , where ( $n > 0$ )

$$\begin{aligned} U_{-n}^T &:= \left(1 + \frac{n-1}{2}\lambda_+\lambda_-\right) \times \\ &\quad \left(a^n, \dots, \binom{n}{k}^{\frac{1}{2}} a^{n-k}c^k, \dots, c^n, a^{n-1}\lambda_+, \dots, \binom{n-1}{k}^{\frac{1}{2}} a^{n-1-k}c^k\lambda_+, \dots, c^{n-1}\lambda_+\right), \\ U_n^T &:= \left(1 + \frac{n+1}{2}\lambda_+\lambda_-\right) \times \\ &\quad \left(d^n, \dots, \binom{n}{k}^{\frac{1}{2}} d^{n-k}b^k, \dots, b^n, d^{n-1}\lambda_-, \dots, \binom{n-1}{k}^{\frac{1}{2}} d^{n-1-k}b^k\lambda_-, \dots, b^{n-1}\lambda_-\right). \end{aligned} \quad (3.32)$$

The matrices  $F_{\pm n}$  are hermitian by construction. To check that they are idempotent, we compute:

$$\begin{aligned} U_{-n}^\dagger U_{-n} &= \left(1 + \frac{n-1}{2}\lambda_+\lambda_-\right)^2 \sum_{k=0}^n (ad)^{n-k}(-bc)^k - \lambda_-\lambda_+ \sum_{k=0}^{n-1} (ad)^{n-1-k}(-bc)^k \\ &= (1 + (n-1)\lambda_+\lambda_-)(ad - bc)^n + \lambda_+\lambda_-(ad - bc)^{n-1} \\ &= (1 + (n-1)\lambda_+\lambda_-)(1 - n\lambda_+\lambda_-) + \lambda_+\lambda_-(1 - (n-1)\lambda_+\lambda_-) \\ &= 1. \end{aligned} \quad (3.33)$$

In the same manner, we check  $U_n^\dagger U_n = 1$ . It remains to verify that the projective modules  $A(S_s^2)^{2n+1}E_{\pm n}$  and  $A(S_s^2)^{2n+1}F_{\pm n}$  are isomorphic. For this purpose, we use (1.26) and take as  $L, \tilde{L}$ , the matrices  $L_{\pm n} := U_{\pm n}Q_{\pm n}^T$ ,  $\tilde{L}_{\pm n} := P_{\pm n}V_{\pm n}^T \in M_{2n+1}(A(S_s^2))$ , respectively. A calculation similar to (3.33) shows that  $Q_{\pm n}^T P_{\pm n} = 1$ . This together with (3.33) and  $U_n^\dagger U_n = 1$  implies that  $L_{\pm n}$  and  $\tilde{L}_{\pm n}$  satisfies (1.27). (Note that  $L_{\pm n}\tilde{L}_{\pm n} = E_{\pm n}$  and  $\tilde{L}_{\pm n}L_{\pm n} = F_{\pm n}$ .) Thus the modules  $A(S_s^2)^{2n+1}E_{\pm n}$  and  $A(S_s^2)^{2n+1}F_{\pm n}$  are isomorphic, as claimed. This hermitian presentation of the projective modules  $A(S_s^3)_{\pm n}$ ,  $n > 0$ , agrees with [L-Ga, (3.25)] for the projectors of the classical Hopf line bundles, and resembles the appropriate formulas obtained in [L-Gb, Section 4.2]. (The case  $n = 0$  is trivial.)

Finally, we want to show that

**Proposition 3.7**  $A(S_s^3)_{-1} \oplus A(S_s^3)_1 = A(S_s^2)^2$  as left  $A(S_s^2)$ -modules.

*Proof.* We can infer from the preceding considerations that the matrix  $\text{diag}(F_{-1}, F_1)$  is a projector matrix of  $A(S_s^3)_{-1} \oplus A(S_s^3)_1$ . First, it turns out technically convenient to conjugate  $F_1$  by

$$M := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.34)$$

Then  $\tilde{F}_1 := MF_1M$  is evidently equivalent (i.e., giving an isomorphic projective module) to  $F_1$  (just take  $L = \tilde{L} = M$  in (1.27)), whence  $\text{diag}(F_{-1}, F_1)$  is equivalent to  $\text{diag}(F_{-1}, \tilde{F}_1)$ . (Note that this way we have  $F_{-1}\tilde{F}_1 = 0 = \tilde{F}_1F_{-1}$ .) To prove the proposition we employ (1.26-1.27), and put  $F = \text{diag}(F_{-1}, \tilde{F}_1)$  and  $E = \text{diag}(1, 1)$ . The point is to find  $L, \tilde{L}$  satisfying (1.27). Since

$$F_{-1} = (a, c, \lambda_+)^T(d, -b, -\lambda_-) \quad \text{and} \quad \tilde{F}_1 = (1 + 2\lambda_+\lambda_-)(b, d, \lambda_-)^T(-c, a, -\lambda_+) , \quad (3.35)$$

we look for  $\tilde{L}$  of the form

$$\tilde{L} = \begin{pmatrix} f_- \\ f_+ \end{pmatrix} , \quad f_- = \begin{pmatrix} a \\ c \\ \lambda_+ \end{pmatrix} (u_+ \quad v_+) , \quad f_+ = \begin{pmatrix} b \\ d \\ \lambda_- \end{pmatrix} (u_- \quad v_-) , \quad (3.36)$$

and for  $L$  of the form

$$L = (g_- \quad g_+) , \quad g_- = \begin{pmatrix} x_- \\ y_- \end{pmatrix} (d \quad -b \quad -\lambda_-) , \quad g_+ = \begin{pmatrix} x_+ \\ y_+ \end{pmatrix} (-c \quad a \quad -\lambda_+) . \quad (3.37)$$

Here to ensure that  $\tilde{L} \in M_{6 \times 2}(A(S_s^2))$  and  $L \in M_{2 \times 6}(A(S_s^2))$  we take  $u_+, v_+, x_+, y_+ \in A(S_s^3)_1$  and  $u_-, v_-, x_-, y_- \in A(S_s^3)_{-1}$ . Using the super determinant relation  $ad - bc + \lambda_+\lambda_- = 1$ , one can verify that

$$u_+ = d, \quad v_+ = -b, \quad x_+ = (1 + 3\lambda_+\lambda_-)b, \quad y_+ = (1 + 3\lambda_+\lambda_-)d, \quad (3.38)$$

$$u_- = -c, \quad v_- = a, \quad x_- = (1 + \lambda_+\lambda_-)a, \quad y_- = (1 + \lambda_+\lambda_-)c, \quad (3.39)$$

is a solution of (1.27), as needed.  $\square$

By analogy with the classical situation, we call  $A(S_s^3)_{-1}$  and  $A(S_s^3)_1$  the super-spin-bundle modules. Proposition 3.7 is a super version of the fact that the module of Dirac spinors, i.e., the direct sum of the spin-bundle modules, is free both for the classical and quantum sphere [LPS]. In fact, the freeness of the module  $P_{-1} \oplus P_1$  [HM99, p.257] of Dirac spinors on the quantum sphere can be shown by precisely the same method as in the super-sphere case. It suffices to take in the proof of Proposition 3.7  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $F_{-1} = (\alpha, \gamma)^T(\delta, -q\beta)$ ,  $F_1 = (\delta, \beta)^T(\alpha, -q^{-1}\gamma)$ ,

$$f_- = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\delta \quad -q\beta) , \quad f_+ = \begin{pmatrix} \beta \\ \delta \end{pmatrix} (-q^{-1}\gamma \quad \alpha) , \quad (3.40)$$

$$g_- = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (\delta \quad -q\beta) , \quad g_+ = \begin{pmatrix} \beta \\ \delta \end{pmatrix} (-q^{-1}\gamma \quad \alpha) , \quad (3.41)$$

where  $\alpha, \beta, \gamma, \delta$  are the generators of  $A(SL_q(2))$  as in [HM99].

## 4 Appendix: Gauge transformations

We follow here the definition of a gauge transformation used in [H-PM96]. For an  $H$ -Galois extension  $B \subseteq P$ , it is defined as a unital convolution-invertible homomorphism  $f : H \rightarrow P$  satisfying  $\Delta_R \circ f = (f \otimes \text{id}) \circ \text{Ad}_R$ . We treat this definition as the first approximation of an appropriate concept of gauge transformations on Hopf-Galois extensions (see [D-Mb] and

the paragraph above Proposition 3.4 in [H-PM96], cf. [D-M97a, Sections 6.1-6.2], [D-M97b], [B-T96, Section 5]). It turns out that the space of strong connections is closed under the action of gauge transformations [H-PM96, Proposition 3.7]. The following theorem describes this action.

**Theorem 4.1** *Let  $B \subseteq P$  be an  $H$ -Galois extension admitting a strong connection. The following describes a left action of gauge transformations on strong connections which is compatible with the identifications of Theorem 2.3:*

- 1)  $(f \triangleright s)(p) := s(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)})$
- 2)  $(f \triangleright D)(p) := D(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)})$
- 3)  $(f \triangleright \Pi)(rdp) := r\Pi(d(p_{(0)}f(p_{(1)})))f^{-1}(p_{(2)}) + rp_{(0)}f(p_{(1)})df^{-1}(p_{(2)})$
- 4)  $(f \triangleright \omega)(h) := f(h_{(1)})\omega(h_{(2)})f^{-1}(h_{(3)}) + f(h_{(1)})d f^{-1}(h_{(2)})$

*Proof.* We need to study the following diagrams:

$$\begin{array}{ccc} GT(P) \times V_i & \xrightarrow{\alpha_i} & V_i \\ \downarrow \text{id} \times J_{ij} & & \downarrow J_{ij} \\ GT(P) \times V_j & \xrightarrow{\alpha_j} & V_j \end{array} \quad (4.1)$$

Here  $\alpha_i$ 's are the corresponding left actions specified above and  $J_{ij}$ 's,  $i, j \in \{1, 2, 3, 4\}$  are obtained in an obvious way by composing suitable bijections  $J_i$  introduced in the proof of Theorem 2.3. We know that  $\alpha_4$  is a well-defined left action [H-PM96, Proposition 3.4]. It suffices to show that

$$\alpha_i = J_{4i} \circ \alpha_4 \circ (\text{id} \times J_{4i}) \quad \text{for } i \in \{1, 2, 3\}. \quad (4.2)$$

For  $i = 3$  it is proved in [H-PM96, Proposition 3.5]. For  $i = 2$ , we have

$$\begin{aligned} & J_{42}(f \triangleright J_{24}(D))(p) \\ &= (J_{12} \circ J_{41})(f \triangleright (J_{34} \circ J_{23})(D))(p) \\ &= 1 \otimes p - J_{41}(f \triangleright (J_{34} \circ J_{23})(D))(p) \\ &= 1 \otimes p - p \otimes 1 - p_{(0)}(f \triangleright (J_{34} \circ J_{23})(D))(p_{(1)}) \\ &= dp - p_{(0)}f(p_{(1)})(J_{34} \circ J_{23})(D)(p_{(2)})f^{-1}(p_{(3)}) - p_{(0)}f(p_{(1)})df^{-1}(p_{(2)}) \\ &= d(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)}) - p_{(0)}f(p_{(1)})p_{(2)}^{[1]}J_{23}(D)(dp_{(2)}^{[2]})f^{-1}(p_{(3)}) . \end{aligned} \quad (4.3)$$

Note now that, since  $P$  admits a strong connection, it is projective (Corollary 2.4) and hence flat as a left  $B$ -module. Consequently  $P \otimes H$  is left  $B$ -flat and

$$\text{Ker}((\Delta_R - \text{id} \otimes 1) \otimes_B \text{id} \otimes \text{id}) = B \otimes_B P \otimes H . \quad (4.4)$$

Using property (1.4) of the translation map and the  $\text{Ad}_R$ -colinearity of  $f$ , we obtain

$$(\Delta_R - \text{id} \otimes 1)(p_{(0)}f(p_{(1)})p_{(2)}^{[1]}) \otimes_B p_{(2)}^{[2]} \otimes p_{(3)} = 0 . \quad (4.5)$$

Hence

$$p_{(0)}f(p_{(1)})p_{(2)}^{[1]} \otimes_B p_{(2)}^{[2]} \otimes p_{(3)} \in B \otimes_B P \otimes H, \quad (4.6)$$

and we have:

$$\begin{aligned}
J_{42}(f \triangleright J_{24}(D))(p) &= d(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)}) - (J_{23}(D) \circ d)(p_{(0)}f(p_{(1)})p_{(2)}^{[1]}p_{(2)}^{[2]})f^{-1}(p_{(3)}) \\
&= d(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)}) + (D - d)(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)}) \\
&= D(p_{(0)}f(p_{(1)}))f^{-1}(p_{(2)}) \\
&= \alpha_2(f, D)(p) .
\end{aligned} \tag{4.7}$$

Similarly, we compute:

$$J_{41}(f \triangleright J_{14}(s))(p) = p \otimes 1 + p_{(0)}f(p_{(1)})J_{14}(s)(p_{(2)})f^{-1}(p_{(3)}) + p_{(0)}f(p_{(1)})df^{-1}(p_{(2)}) . \tag{4.8}$$

On the other hand,

$$\begin{aligned}
J_{14}(s)(h) &= (J_{34} \circ J_{23} \circ J_{12})(s)(h) \\
&= h^{[1]}(J_{23} \circ J_{12})(dh^{[2]}) \\
&= h^{[1]}(d - J_{12}(s))(h^{[2]}) \\
&= h^{[1]}(s - \text{id} \otimes 1)(h^{[2]}) \\
&= h^{[1]}s(h^{[2]}) - \varepsilon(h) \otimes 1 .
\end{aligned} \tag{4.9}$$

Therefore, taking advantage of the left  $B$ -linearity of  $s$ , (4.6) and (1.6), we obtain

$$\begin{aligned}
&J_{41}(f \triangleright J_{14}(s))(p) \\
&= p_{(0)}f(p_{(1)}) \otimes f^{-1}(p_{(2)}) + p_{(0)}f(p_{(1)})p_{(2)}^{[1]}s(p_{(2)}^{[2]})f^{-1}(p_{(3)}) - p_{(0)}f(p_{(1)}) \otimes f^{-1}(p_{(2)}) \\
&= s(p_{(0)}f(p_{(1)}) \otimes f^{-1}(p_{(2)})) \\
&= \alpha_1(f, s)(p) ,
\end{aligned} \tag{4.10}$$

as needed.  $\square$

*Remark 4.2* The gauge transformations on  $H$ -Galois extension  $B \subseteq P$  are in on-to-one correspondence with the gauge automorphisms understood as unital left  $B$ -linear right  $H$ -colinear automorphisms of  $P$  [B-T96, Proposition 5.2]. If  $f : H \rightarrow P$  is a gauge transformation, then  $F : P \rightarrow P$ ,  $F(p) := p_{(0)}f(p_{(1)})$  is a gauge automorphism. Analogously, for  $\alpha \in \Omega^1 P$ , we put  $F(\alpha) := ((\text{id} \otimes m) \circ (\text{id} \otimes \text{id} \otimes f) \circ \Delta_{\Omega^1 P})(\alpha)$ . (The other way round we have  $f(h) = h^{[1]}F(h^{[2]})$ .) Due to the right  $H$ -colinearity of the covariant differential  $D$ , we can re-write point 2) of the above theorem as  $(D \triangleleft F)(p) = F^{-1}(DF(p))$ . This formula coincides with the usual formula for the action of gauge transformations on projective-module connections (e.g., see [C-A94, p.554]).  $\diamond$

*Remark 4.3* In the sense of the definition considered here, the connections in Example 2.15 are not gauge equivalent. This is because, for the quantum Hopf fibration, any gauge transformation  $f$  acts trivially on the space of connections. Indeed, since  $H$  is spanned by group-like elements,  $f$  is convolution-invertible, and the only invertible elements in  $A(SL_q(2))$  are non-zero complex numbers [HM99, Appendix],  $f$  must be  $\mathbb{C} \setminus \{0\}$ -valued. This effect is due to working with non-completed (polynomial) algebras.  $\diamond$

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